УДК 515.12

К ТЕОРИИ НЕПРЕРЫВНЫХ ГОМОМОРФИЗМОВ

Т.Дж. Касымова

Для непрерывных гомоморфизмов топологических групп введено понятие базы, посредством которого исследована параллельность (вложение) непрерывных гомоморфизмов в категории *GTOP(H)*.

Ключевые слова: база непрерывного гомоморфизма; категория; характер.

TO THE THEORY OF CONTINUOUS HOMOMORPHISMS

T.J. Kasymova

For continuous homomorphisms of topological groups a concept of base is introduced and the parallel-ability (inclusion) of continuous homomorphisms is researched by means of it in the category GTOP(H).

Keywords: a base of continuous homomorphism; category; character.

An idea to transfer some concepts and statements concerning spaces to mappings allows to generalize many results. In this way works of L.S. Pontryagin [1], B.A. Pasynkov [2], A.A. Borubaev [3], A.A. Chekeev [4] and others are known. A transferring of these results to algebraic objects and studying of their behavior is the actual task significantly enriching the theory of uniform spaces and as a result the theory of uniformly continuous mappings. So, for example, the topological group has algebraic structure, on the one hand, and is a topological space on another one. Below the concept of base of uniformly continuous mapping ([3]) is postponed for continuous homomorphisms of topological groups.

Remind of some basic concepts from books [5], [6].

A family $B(x)$ of neighborhoods of x is called a *base for a topological space X at the point x* or a *local base* if for any neighborhood V of x there exists a $U \in B(x)$ such that $x \in U \subset V$. If B is a base for X then the family $B(x)$ consisting of all elements of B that contain x is a base for X at the point x. On the other hand, if for every $x \in X$ a base for X at the point x is given then the union $B=\cup\{B(x):x\in X\}$ is a base for X.

The *character of a point* $x\in X$ is defined as the smallest cardinal number of form $|B(x)|$, where $B(x)$ is a base for a topological space X at the point x; $|\bullet|$ stands for cardinality; this cardinal number is denoted by $\chi(x, X)$. The *character of a topological space X* is defined as the supremum of all numbers $\chi(x, X)$ for $x\in X$, i.e. $\chi(X)=sup\{\chi(x, X):x\in X\}$.

Definition [6]. A set G allocated with structures of group and topology is called a *topological group* if it satisfies the following two axioms:

(GT_I) Mapping $(x, y)\mapsto xy$ o|f product $G\times G$ into G is continuous.

(GT_{II}) Mapping $x\mapsto x^{-1}$ of group G into itself (symmetry of group G) is continuous.

Axioms (GT_I) and (GT_{II}) are equivalent to the next axiom:

(GT) Mapping $(x, y)\mapsto xy^{-1}$ of product $G\times G$ into G is continuous.

Every topological group has a base $B(e)$ of *neighborhoods filters of unit,* satisfying to the following axioms:

(GV_I) For any $U \in B(e)$ there exists $V \in B(e)$ such that $VV\subset U$.

(GV_{II}) For any $U \in B(e)$ there exists $V \in B(e)$ such that $V^{-1} \subset U$.

(GV_{III}) For any $a \in G$ and $U \in B(e)$ there exists $V \in B(e)$ is containing into aUa^{-1}.

Axioms (GV_I) and (GV_{II}) can be reformulated as:

(GV) For any $U \in B(e)$ there exists $V \in B(e)$ such that $VV^{-1} \subset U$.

Let $f:G\to H$ be a continuous homomorphism of topological group G into topological group H, $B(e_G)$, $B(e)$ are a bases of neighborhoods filters of units $e_G \in G$ and $e\in H$, respectively, $B_f(e_G)$ be a neighborhoods system, generated group topology on G, generally speaking, more weak, than initial one, i.e. $B_f(e_G) \subseteq B(e_G)$.

Definition 1. A neighborhoods system $B_f(e_G)$ is said to be a *base of continuous homomorphism f*, if for any neighborhood $U \in B(e_G)$ there exist such neighborhoods $V \in B(e)$ and $W \in B_f(e_G)$ that $f^{-1}(V) \cap W \subset U$ holds and a character $\chi(f) \leq \tau$, if $|B_f(e_G)| \leq \tau$.

It is known [7], that objects in category *GTOP* are all separated topological groups, and morphisms are continuous homomorphisms.

We denote a *category of all continuous homomorphisms* as *GTOP(H)* it's an objects are continuous homomorphisms $f{:}G \to H$, $g{:}N \to H$ *with fixed bases* $B_f(e_G)$ and $B_g(e_N)$ respectively, and morphism from an object $f \in GTOP(H)$ into object $g \in GTOP(H)$ is called *homomorphism* $h{:}G \to N$ *continuous* with respect to topologies, induced by bases $B_f(e_G)$ and $B_g(e_N)$ such that $f = g \cdot h$.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "h"'] & & H \\ & N \arrow[ur, "g"'] & \end{tikzcd}$$

.

In this case we write $h{:}f \to g$.

Lemma. If $B_f(e_G)$ is a base of object $f{:}G \to H$ of category *GTOP(H)*, then a family $N = \cap\{W : W \in B_f(e_G)\}$ is a normal subgroup of group G.

Proof. Let G be a topological group having a base $B(e_G)$ of neighborhoods filters of unit $e_G \in G$, $B_f(e_G)$ be a neighborhoods system, generated group topology on G, generally speaking, more weak, than initial one, i. e. $B_f(e_G) \subseteq B(e_G)$. Denote as $N = \cap\{W : W \in B_f(e_G)\}$ a intersection of all neighborhoods of the base $B_f(e_G)$ of continuous homomorphism $f{:}G \to H$ (according to Definition 1).

We show that N is a subgroup of group G. By construction $N \subset W$ for any $W \in B_f(e_G)$. On axioms (GV) of base of filter of unit and (GT) of topological group we have:

1) for any $W \in B_f(e_G)$ there exists such $V \in B_f(e_G)$ that $VV^{-1} \subset W$, moreover $NN^{-1} \subset W$ for any $W \in B_f(e_G)$, it means $NN^{-1} \subset N$, i.e. N is a subgroup of group G.

2) For any $W \in B_f(e_G)$ and $a \in N$ there exists such $W \in B_f(e_G)$ is containing into aWa^{-1}, i. e. $V \subset aWa^{-1}$ and $N \subset aWa^{-1}$. Then $a^{-1}Na \subset W$, hence $aNa^{-1} \subset N$, i.e. N is a normal subgroup of group G.

The concept of the mapping parallel to space was introduced by B. A. Pasynkov [2] for topological spaces, it was done by A.A. Borubaev [3] for uniform spaces.

We introduce this concept for continuous homomorphisms of topological groups.

Definition 2. Let $f{:}G \to H$ be a continuous homomorphism of topological group G into topological group H, $\pi : G \to G/N$ be a projection of G onto factorgroup G/N on the normal subgroup N. A continuous homomorphism *f is parallel* to projection π (is denoted as $f \| \pi$) if there exists such continuous mapping $i : G \to G/N \times H$ of topological group G into topological group $G/N \times H$, that $i = f \Delta \pi$ and $f = p_H|_{i(G)} : G \to H$. So the mapping i is called *inclusion*.

Theorem 1. Let $f \in GTOP(H)$, where $f{:}G \to H$ is continuous homomorphism – an object of category *GTOP(H)*, $\chi(f) \leq \tau$. Then there exists such normal subgroup $N \subset G$ that $f \| p_H$, where $p_H : G/N \times H \to H$.

Proof. Let $B_f(e_G)$ be a base of continuous homomorphism $f : G \to H$ – an object of category *GTOP(H)*, and $|B_f(e_G)| \leq \tau$. There exists natural homomorphism $\pi : G \to G/N$ is a projection of group G onto factorgroup G/N on normal subgroup $N = \bigcap\{W : W \in B_f(e_G)\}$ (according to Lemma), given on rule $\pi(x) = xN$ for any $x \in G$.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, "f\Delta\pi"] & H \\ G/N & G/N \times H \arrow[u, "p_H"'] \end{tikzcd}$$

.

Let us consider diagonal product $i = f \Delta \pi$. It is continuous as diagonal product of continuous homomorphisms. We prove, that $f \Delta \pi$ is inclusion of continuous homomorphism f into projection p_H. Note, that $f = p_H \cdot (f \Delta \pi)$.

We show, that $f \Delta \pi$ is one-to-one. Let $x_1, x_2 \in G$ and $x_1 \neq x_2$. Пусть $f(x_i) \in V_i$, $V_i \in B(e)$, $i = 1,2$ of group H. Then there exist $W_i \in B_f(e_G)$ such that $x_i \in (f \vartriangle \pi)^{-1}(W_i) \subset U_i$, $i = 1,2$. We have

$((f\Delta\pi)^{-1}(\mathrm{W}_1)\cap f^{-1}(\mathrm{V}_1))\cap((f\Delta\pi)^{-1}(\mathrm{W}_2)\cap f^{-1}(\mathrm{V}_2))=\varnothing$, hence $(\mathrm{W}_1\times\mathrm{V}_1)\cap(\mathrm{W}_2\times\mathrm{V}_2)=\varnothing$.

Then, on properties of neighborhoods system [5], $\mathrm{U}\subset\mathrm{U}_1\cap\mathrm{U}_2\in B(e_G)$ and $\mathrm{V}\subset\mathrm{V}_1\cap\mathrm{V}_2\in B(e)$ such that $(\mathrm{U}\cap f^{-1}(\mathrm{V}))(x_1)\cap(\mathrm{U}\cap f^{-1}(\mathrm{V}))(x_2)=\varnothing$. Therefore, there exists $\mathrm{W}\subset\mathrm{W}_1\cap\mathrm{W}_2\in B_f(e_G)\subseteq B(e_G)$ such that $(f\Delta\pi)^{-1}(\mathrm{W})\subset\mathrm{U}$. It means that $(f\Delta\pi)(x_1)\neq(f\Delta\pi)(x_2)$, i. e. $f\Delta\pi$ is isomorphism.

As topological group G is isomorphically enclosed into $G/N\times H$, then $(f\Delta\pi)(\mathrm{U})\in\mathrm{W}\times\mathrm{V}|_{(f\Delta\pi)(G)}$ for any neighborhood $\mathrm{U}\in B(e_G)$. Then on Definition 1 $\mathrm{W}\cap f^{-1}(\mathrm{V})\subset\mathrm{U}$, hence $(\mathrm{W}\times\mathrm{V})\cap(f\Delta\pi)(G)\subset$. $(f\Delta\pi)(\mathrm{U})$ The projection $p_H: G/N\times H\to H$ of Cartesian product of G/N and topological group H on factor H is given by $(\pi(x),f(x))=p_H(x)$ for any $x\in G$. So the diagonal product $f\Delta\pi$ is inclusion of f into p_H, i.e. $f\|p_H$.

References

1. *Pontryagin L.S.* Continuous groups / L.S. Pontryagin. Moscow: Fizmatgiz, 1973. 527 p. (in Russian).
2. *Pasynkov B.A*. On transferring on mappings of some concepts and statements concerning spaces / B.A. Pasynkov // Mappings and functors. Moscow: MSU, 1984. P. 72–102. (in Russian).
3. *Borubaev A.A*. Uniform spaces and uniformly continuous mappings / A.A. Borubaev. Bishkek, 1990. 170 p. (in Russian).
4. *Borubaev A.A.* Uniform structures on the topological spaces and groups /A.A. Borubaev, A.A. Chekeev. KSPU named after I. Arabaev. Bishkek, 1997. 267 p. (in Russian).
5. *Engelking R*. General topology / R. Engelking. Moscow: Mir, 1986. 515 p. (in Russian).
6. *Bourbaki N.* General topology. Topological groups / N. Bourbaki. Moscow: Nauka, 1969. 392 p. (in Russian).
7. *Bucur I.* Introduction to the theory of categories and functors / I. Bucur, A. Deleanu. Moscow: Mir, 1972. 259 p. (in Russian).