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К ТЕОРИИ НЕПРЕРЫВНЫХ ГОМОМОРФИЗМОВ

Т.Дж. Касимова

Для непрерывных гомоморфизмов топологических групп введено понятие базы, посредством которого исследована параллельность (вложение) непрерывных гомоморфизмов в категории $GTOP(H)$.

Ключевые слова: база непрерывного гомоморфизма; категория; характер.

TO THE THEORY OF CONTINUOUS HOMOMORPHISMS

T.J. Kasymova

For continuous homomorphisms of topological groups a concept of base is introduced and the parallel-ability (inclusion) of continuous homomorphisms is researched by means of it in the category $GTOP(H)$.

Keywords: a base of continuous homomorphism; category; character.

An idea to transfer some concepts and statements concerning spaces to mappings allows to generalize many results. In this way works of L.S. Pontryagin [1], B.A. Pasyukov [2], A.A. Borubaev [3], A.A. Chekeev [4] and others are known. A transferring of these results to algebraic objects and studying of their behavior is the actual task significantly enriching the theory of uniform spaces and as a result the theory of uniformly continuous mappings. So, for example, the topological group has algebraic structure, on the one hand, and is a topological space on another one. Below the concept of base of uniformly continuous mapping ([3]) is postponed for continuous homomorphisms of topological groups.

Remind of some basic concepts from books [5], [6].

A family $B(x)$ of neighborhoods of x is called a *base for a topological space X at the point x* or a *local base* if for any neighborhood V of x there exists a $U \in B(x)$ such that $x \in U \subset V$. If B is a base for X then the family $B(x)$ consisting of all elements of B that contain x is a base for X at the point x . On the other hand, if for every $x \in X$ a base for X at the point x is given then the union $B = \cup \{B(x) : x \in X\}$ is a base for X .

The *character of a point $x \in X$* is defined as the smallest cardinal number of form $|B(x)|$, where $B(x)$ is a base for a topological space X at the point x ; $|\cdot|$ stands for cardinality; this cardinal number is denoted by $\chi(x, X)$. The *character of a topological space X* is defined as the supremum of all numbers $\chi(x, X)$ for $x \in X$, i.e. $\chi(X) = \sup \{\chi(x, X) : x \in X\}$.

Definition [6]. A set G allocated with structures of group and topology is called a *topological group* if it satisfies the following two axioms:

(GT_I) Mapping $(x, y) \mapsto xy$ of product $G \times G$ into G is continuous.

(GT_{II}) Mapping $x \mapsto x^{-1}$ of group G into itself (symmetry of group G) is continuous.

Axioms (GT_I) and (GT_{II}) are equivalent to the next axiom:

(GT) Mapping $(x, y) \mapsto xy^{-1}$ of product $G \times G$ into G is continuous.

Every topological group has a base $B(e)$ of *neighborhoods filters of unit*, satisfying to the following axioms:

(GV_I) For any $U \in B(e)$ there exists $V \in B(e)$ such that $VV \subset U$.

(GV_{II}) For any $U \in B(e)$ there exists $V \in B(e)$ such that $V^{-1} \subset U$.

(GV_{III}) For any $a \in G$ and $U \in B(e)$ there exists $V \in B(e)$ is containing into aUa^{-1} .

Axioms (GV_I) and (GV_{II}) can be reformulated as:

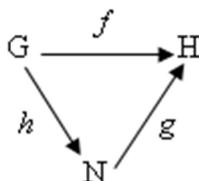
(GV) For any $U \in B(e)$ there exists $V \in B(e)$ such that $VV^{-1} \subset U$.

Let $f: G \rightarrow H$ be a continuous homomorphism of topological group G into topological group H , $B(e_G)$, $B(e)$ are a bases of neighborhoods filters of units $e_G \in G$ and $e \in H$, respectively, $B_f(e_G)$ be a neighborhoods system, generated group topology on G , generally speaking, more weak, than initial one, i.e. $B_f(e_G) \subseteq B(e_G)$.

Definition 1. A neighborhoods system $B_f(e_G)$ is said to be a *base of continuous homomorphism* f , if for any neighborhood $U \in B(e_G)$ there exist such neighborhoods $V \in B(e)$ and $W \in B_f(e_G)$ that $f^{-1}(V) \cap W \subset U$ holds and a character $\chi(f) \leq \tau$, if $|B_f(e_G)| \leq \tau$.

It is known [7], that objects in category $GTOP$ are all separated topological groups, and morphisms are continuous homomorphisms.

We denote a *category of all continuous homomorphisms* as $GTOP(H)$ it's an objects are continuous homomorphisms $f:G \rightarrow H$, $g:N \rightarrow H$ with fixed bases $B_f(e_G)$ and $B_g(e_N)$ respectively, and morphism from an object $f \in GTOP(H)$ into object $g \in GTOP(H)$ is called *homomorphism $h:G \rightarrow N$ continuous* with respect to topologies, induced by bases $B_f(e_G)$ and $B_g(e_N)$ such that $f = g \cdot h$.



In this case we write $h:f \rightarrow g$.

Lemma. If $B_f(e_G)$ is a base of object $f:G \rightarrow H$ of category $GTOP(H)$, then a family $N = \bigcap \{W:W \in B_f(e_G)\}$ is a normal subgroup of group G .

Proof. Let G be a topological group having a base $B(e_G)$ of neighborhoods filters of unit $e_G \in G$, $B_f(e_G)$ be a neighborhoods system, generated group topology on G , generally speaking, more weak, than initial one, i. e. $B_f(e_G) \subseteq B(e_G)$. Denote as $N = \bigcap \{W:W \in B_f(e_G)\}$ a intersection of all neighborhoods of the base $B_f(e_G)$ of continuous homomorphism $f:G \rightarrow H$ (according to Definition 1).

We show that N is a subgroup of group G . By construction $N \subset W$ for any $W \in B_f(e_G)$. On axioms (GV) of base of filter of unit and (GT) of topological group we have:

1) for any $W \in B_f(e_G)$ there exists such $V \in B_f(e_G)$ that $VV^{-1} \subset W$, moreover $NN^{-1} \subset W$ for any $W \in B_f(e_G)$, it means $NN^{-1} \subset N$, i.e. N is a subgroup of group G .

2) For any $W \in B_f(e_G)$ and $a \in N$ there exists such $V \in B_f(e_G)$ is containing into aWa^{-1} , i. e. $V \subset aWa^{-1}$ and $N \subset aWa^{-1}$. Then $a^{-1}Na \subset W$, hence $aNa^{-1} \subset N$, i.e. N is a normal subgroup of group G .

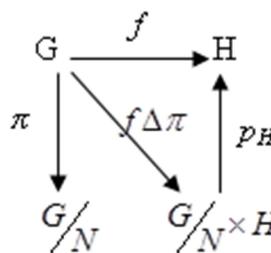
The concept of the mapping parallel to space was introduced by B. A. Pasynkov [2] for topological spaces, it was done by A.A. Borubaev [3] for uniform spaces.

We introduce this concept for continuous homomorphisms of topological groups.

Definition 2. Let $f:G \rightarrow H$ be a continuous homomorphism of topological group G into topological group H , $\pi:G \rightarrow G/N$ be a projection of G onto factorgroup G/N on the normal subgroup N . A continuous homomorphism f is *parallel* to projection π (is denoted as $f \parallel \pi$) if there exists such continuous mapping $i:G \rightarrow G/N \times H$ of topological group G into topological group $G/N \times H$, that $i = f \Delta \pi$ and $f = p_H \circ i|_{i(G)}$. So the mapping i is called *inclusion*.

Theorem 1. Let $f \in GTOP(H)$, where $f:G \rightarrow H$ is continuous homomorphism – an object of category $GTOP(H)$, $\chi(f) \leq \tau$. Then there exists such normal subgroup $N \subset G$ that $f \parallel p_H$, where $p_H:G/N \times H \rightarrow H$.

Proof. Let $B_f(e_G)$ be a base of continuous homomorphism $f:G \rightarrow H$ – an object of category $GTOP(H)$, and $|B_f(e_G)| \leq \tau$. There exists natural homomorphism $\pi:G \rightarrow G/N$ is a projection of group G onto factorgroup G/N on normal subgroup $N = \bigcap \{W:W \in B_f(e_G)\}$ (according to Lemma), given on rule $\pi(x) = xN$ for any $x \in G$.



Let us consider diagonal product $i = f \Delta \pi$. It is continuous as diagonal product of continuous homomorphisms. We prove, that $f \Delta \pi$ is inclusion of continuous homomorphism f into projection p_H . Note, that $f = p_H \cdot (f \Delta \pi)$.

We show, that $f \Delta \pi$ is one-to-one. Let $x_1, x_2 \in G$ and $x_1 \neq x_2$. Пусть $f(x_i) \in V_i$, $V_i \in B(e)$, $i=1,2$ of group H . Then there exist $W_i \in B_f(e_G)$ such that $x_i \in (f \Delta \pi)^{-1}(W_i) \subset U_i$, $i=1,2$. We have

$((f\Delta\pi)^{-1}(W_1) \cap f^{-1}(V_1)) \cap ((f\Delta\pi)^{-1}(W_2) \cap f^{-1}(V_2)) = \emptyset$,
hence $(W_1 \times V_1) \cap (W_2 \times V_2) = \emptyset$.

Then, on properties of neighborhoods system [5], $U \subset U_1 \cap U_2 \in B(e_G)$ and $V \subset V_1 \cap V_2 \in B(e)$ such that $(U \cap f^{-1}(V))(x_1) \cap (U \cap f^{-1}(V))(x_2) = \emptyset$. Therefore, there exists $W \subset W_1 \cap W_2 \in B_f(e_G) \subseteq B(e_G)$ such that $(f\Delta\pi)^{-1}(W) \subset U$. It means that $(f\Delta\pi)(x_1) \neq (f\Delta\pi)(x_2)$, i. e. $f\Delta\pi$ is isomorphism.

As topological group G is isomorphically enclosed into $G/N \times H$, then $(f\Delta\pi)(U) \in W \times V|_{(f\Delta\pi)(G)}$ for any neighborhood $U \in B(e_G)$. Then on Definition 1 $W \cap f^{-1}(V) \subset U$, hence $(W \times V) \cap (f\Delta\pi)(G) \subset (f\Delta\pi)(U)$. The projection $p_H : G/N \times H \rightarrow H$ of Cartesian product of G/N and topological group H on factor H is given by $(\pi(x), f(x)) = p_H(x)$ for any $x \in G$. So the diagonal product $f\Delta\pi$ is inclusion of f into p_H , i.e. $f \parallel p_H$.

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