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ОБ u-СОВЕРШЕННЫХ ОТОБРАЖЕНИЯХ

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Получены внутренняя и категорная характеристики *u*-непрерывных отображений в терминах декартовых квадратов.

Ключевые слова: u-непрерывное отображение; *u*-совершенное отображение; категория; zu-ультрафильтр; сходимость.

ON u-PERFECT MAPPINGS

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The inner and categorical characterizations of u-continuous mappings in pullback squares terms have been obtained.

Keywords: u-continuous mapping; u-perfect mapping; category; z_u-ultrafilter; convergence.

Basic features of uniform topology are presented in the books [1–3]. Every uniform space is denoted as *uX* where *X* is a Tychonoff space and *u* is a uniformity on *X* in uniform coverings terms [1, 2]. We denote the set of all (bounded) uniformly continuous functions on *uX* by $C(uX)$ (by $C^*(uX)$) and $\mathfrak{Z}_u = \{f^{-1}(0) : f \in C(uX)\} = \mathfrak{Z}_u^* = \{g^{-1}(0) : g \in C^*(uX)\}$, since $\min\{|f|,1\} \in C^*(uX)$ for any $f \in C(uX)$.

The maximal centered systems of elements $\mathfrak{Z}_u$ are called z_u–ultrafilters. In [4] a mapping $f : uX \to vY$ of uniform space *uX* into uniform space *vY* is called *u*–continuous, if $f^{-1}(Z) \in \mathfrak{Z}_u$, $f^{-1}(Y \setminus Z) \in C\mathfrak{Z}_u = \{X \setminus N : N \in \mathfrak{Z}_u\}$ for any $Z \in \mathfrak{Z}_u$. In [4] it is proved that for uniformly continuous functions $h_i : uX \to I = [0,1], i = 1,2$ such that $h_1^{-1}(0) \cap h_2^{-1}(0) = \varnothing$ function $f : uX \to I$, determined as $f(x) = h_1(x) / (h_1(x) + h_2(x))$ for any $x \in X$, is $u-$continuous, and generally speaking, is not uniformly continuous. In [5], [6] important properties of *u*–closed mappings are determined and established, and in [5] also $u-$perfect mappings are introduced.

Definition 1. A mapping $f : uX \to vY$ of uniform space *uX* into uniform space vY is called *u-perfect* if:

1) it is *u*-continuous; 2) it is closed; 3) it is bicompact, i.e. $f^{-1}(y)-$bicompact in *X* for any point $y \in Y$.

Theorem 2. *A uniform space uX is bicompact if and only if every* z_u-*ultrafilter is converging in uX.*

Proof. If uniform space *uX* is bicompact, then all ultrafilters are converging in it ([3]), in particularly, all z_u–ultrafilters are converging.

Conversely, let *F* be an arbitrary centered system of closed sets in uniform space *uX*. As $\mathfrak{Z}(uX)$ is a closed sets base of *uX*, then for any $F \in F$ there is such family $\xi_F \subset \mathfrak{Z}(uX)$, that $F = \cap \xi_F$. Then family $\xi = \{\xi_F : F \in F\} \subset \mathfrak{Z}(uX)$ is centered. Let p_ξ be such z_u–ultrafilter, that $\xi \subset p_\xi$. Then $\cap p_\xi = \{x\}$ for some point $x \in X$ and $\{x\} = \cap p_\xi \subset \cap \xi \subset \cap F$, i.e. $\cap F \neq \varnothing$. Therefore uniform space *uX* is bicompact.

It is known Franklin [7] and Herrlich [8] established the characterization of perfect mappings by means of Stone-Čech compactification in category *Tych* of Tychonoff spaces and its continuous mappings, and Borubaev

[1] established the characterization of uniformly perfect mappings by means of Samuel compactification in category *Unif* of uniform spaces and its uniformly continuous mappings.

We denote as *ZUnif* the category, whose objects are a uniform spaces, and morphisms are *u*–continuous mappings. In this category in pullback squares terms by means of compactification $\beta_u X$ [9] of uniform space uX the characterization of *u*–perfect mappings of uniform spaces have been obtained.

Theorem 3. *Let uX and vY be a uniform spaces. Then for u–continuous mapping* $f: uX \to vY$ *the next conditions are equivalent:*

(1) f *is u–perfect.*

(2) *If* p *is* $z_u -$*ultrafilter on uX and prefilter* $f(p)=\{f(Z): Z \in p\}$ *is converging to point* $y \in Y$*, then p is converging to point* $x \in f^{-1}(y)$.

(3) *For extension mapping* $\beta f: \beta_u X \to \beta_v Y$ *a remainder* $\beta_u X \setminus X$ *transfers to a remainder* $\beta_v Y \setminus Y$*, i.e.* $\beta f(\beta_u X \setminus X) \subset \beta_v Y \setminus Y$.

(4) *Square*
$$\begin{array}{ccc} uX & \xrightarrow{i_X} & \beta_u X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta f} \\ vY & \xrightarrow[i_Y]{} & \beta_v Y \end{array} \qquad (*)$$
is pullback in category $ZUnif$.

Proof. $(1) \Rightarrow (2)$. Let f be *u*–perfect mapping and p such z_u–ultrafilter on *uX*, that prefilter $f(p)$ is converging to point $y \in Y$. A set of all uniformly closed sets $Q \in \mathfrak{Z}(vY)$, which are neighborhoods of point y, forms z_v–ultrafilter q on uniform space *vY*. A mapping $f: uX \to vY$ is *u*–continuous, so $f^{-1}(Q) \in p$ for any $Q \in q$ and we have $f^{-1}(y) = \cap\{f^{-1}(Q): Q \in q\} = f^{-1}(\cap q)$. If z_u–ultrafilter p is converging, then it is converging to some point $f^{-1}(y)$ of inverse image. We suppose, that $z_u -$ ultrafilter is not converging. Then for any point $x \in f^{-1}(y)$ there is such $V_x \in L(uX)$ and $Z_x \in \mathfrak{Z}(uX)$, that $x \in V_x \subset [V_i]_X \subset Z_x$ and $Z_x \notin p$ ([4]). A family $\{V_x : x \in f^{-1}(y)\}$ is open covering of bicompact $f^{-1}(y)$. Let $\{V_{x_i} : i = 1, 2, ..., n\}$ be a finite subcovering. Then $\bigcup_{i=1}^{n} Z_i \notin p$, $V = \bigcup_{i=1}^{n} V_{x_i} \in L(uX)$ hence $X \setminus V \in \mathfrak{Z}(uX)$. Since $X \setminus V \cup \bigcup_{i=1}^{n} Z_i = X$ then $X \setminus V \in p$. Then $f(X \setminus V)$ is closed and $f(X \setminus V) \in f(p)$. A set $Y \setminus f(X \setminus V)$ is open neighborhood of point *y*. Therefore there is such $Q' \in q$ that $y \in Q' \subset Y \setminus f(X \setminus V)$. Then $Q' \cap f(X \setminus V) = \varnothing$, so $f^{-1}(Q') \cap X \setminus V = \varnothing$ is contradiction, as $f^{-1}(Q') \in p$ and $X \setminus V \in p$.

$(2) \Rightarrow (1)$. Let p' be an arbitrary $z_{u'} -$ultrafilter in uniform subspace $u'f^{-1}(y)$, where and $y \in Y$ be an arbitrary point. From properties of uniformly closed sets ([4]) it follows, that for any $Z' \in p'$ there is such $Z \in \mathfrak{Z}(uX)$, that $Z' = Z \cap f^{-1}(y)$. Let $\xi_{p'} = \{Z \in \mathfrak{Z}(uX): Z' = Z \cap f^{-1}(y) \text{ and } Z' \in p'\}$. Then there is such $z_u -$ultrafilter p on uX, that $\xi_{p'} \subset p$. Let q be $z_v -$ultrafilter in vY, consisting of all uniformly closed neighborhoods of point *y*. Then $\cap q = \cap\{Q : Q \in q\} = \{y\}$, $Q \in \mathfrak{Z}(vY)$ and $f^{-1}(Q) \in p$ for any $Q \in q$. Then prefilter $f(p)$ is converging to point $y \in Y$. So $z_u -$ultrafilter p is converging to some point $x \in f^{-1}(y)$. Then $z_{u'} -$ultrafilter p' is converging to point $x \in f^{-1}(y)$, hence on Theorem 2 a uniform space $u'f^{-1}(y)$ is bicompact for any point $y \in Y$.

We show a closeness of mapping *f*. Let $F \subset X$ be closed set and $y \in [f(F)]_Y$ be an arbitrary point. Let *q* be z_u – ultrafilter consisting of all uniformly closed neighborhoods of point $y \in Y$. Then $Q \cap f(F) \neq \varnothing$ for any $Q \in q$. Therefore $f(f^{-1}(Q) \cap F) = Q \cap f(X) \neq \varnothing$. For every $Q \in q$, $f^{-1}(Q) \in \mathfrak{Z}(uX)$ and family $\{f^{-1}(Q) \cap F : Q \in q\}$ is centered and $f^{-1}(Q) \cap F \in \mathfrak{Z}(u'F)$, where $u' = u \wedge F$ ([8], [7]) containing centered family $\{f^{-1}(Q) \cap F : Q \in q\}$, and *p* such z_u – ultrafilter in uniform space uX that $\xi_{p'} \subset p$, where $\xi_{p'} = \{Z \in \mathfrak{Z}(uX) : Z' = Z \cap f^{-1}(y) \ u \ Z' \in p'\}$. A prefilter $f(p')$ is converging to y and since $f^{-1}(Q) \in p$ for any $Q \in q$ then $f(p)$ is converging to *y* too. Then on condition of theorem z_u – ultrafilter *p* is converging to some point $x \in f^{-1}(y)$. Then $z_{u'}$ – ultrafilter p' also is converging to point $x \in f^{-1}(y)$. Since *F* is closed in *X*, then $x \in F$. So we have $y = f(x) \in f(F)$, i.e. $[f(F)]_Y \subset f(F)$. Obviously the inverse inclusion $f([F]_X) = f(F) \subset [f(F)]_Y$. Thus $f(F) = [f(F)]_Y$ and *f* is closed mapping.

$(2) \Rightarrow (3)$. Let $x \in \beta_u X \setminus X$ be an arbitrary point. Then there is unique z_u – ultrafilter *p* on uX such that $\{x\} = \cap\{[Z]_{\beta_u X} : Z \in p\}$. For extension mapping $\beta_u f : \beta_u X \to \beta_v Y$ the equality $\beta_u f([Z]_{\beta_u X}) = [f(Z)]_{\beta_v Y}$ holds for any $Z \in p$. Then $\beta_u f(x) = \cap\{[f(Z)]_{\beta_u X} : Z \in p\} = \{y\}$ for some point $y \in \beta_v Y$. Suppose that $y \in Y$. Then prefilter $f(p) = \{f(Z) : Z \in p\}$ is converging to *y* and on condition of theorem z_u – ultrafilter *p* is converging to some point $x' \in f^{-1}(y)$. Evidently, that $\{x'\} = \cap\{[Z]_{\beta_u X} : Z \in p\}$, i.e. $x = x'$ is contradiction. Therefore $y = f(x) \in \beta_v Y \setminus Y$.

$(3) \Rightarrow (2)$. Let *p* be an arbitrary z_u – ultrafilter in uX and prefilter $f(p) = \{f(Z) : Z \in p\}$ is converging to point $y \in Y$. On property (4) of theorem 2.4. ([9]) we have $\{x\} = \cap\{[Z]_{\beta_u X} : Z \in p\} \in \beta_u X$ and point *x* is unique. Then $\beta_u f([Z]_{\beta_u X}) = [f(Z)]_{\beta_v Y}$ for any $Z \in p$ and $\beta_u f(x) = \beta_u f(\cap\{[Z]_{\beta_u X} : Z \in p\}) =$ $= \cap\{[f(Z)]_{\beta_v Y} : Z \in p\} = y$, i.e. $x \in (\beta_u f)^{-1}(y)$.

As $\beta_u f(\beta_u X \setminus X) \subset \beta_v Y \setminus Y$ then $x \in X$.

$(3) \Rightarrow (4)$. We suppose, that for some object *WZ* of category *ZUnif*, $h : \mathrm{WZ} \to \beta_u X$ and $g : \mathrm{WZ} \to vY$ are such *u*–continuous mappings, that $\beta_u f \circ h = i_Y \circ g$. Since $(i_Y \circ g)(Z)$ are containing in $\beta_v Y$ and $\beta_u f(\beta_u X \setminus X) \subset \beta_v Y \setminus Y$, then $h(Z) \subset X$. We determine a mapping $h' : \mathrm{WZ} \to uX$ as $h'(z) = h(z)$ for any $z \in Z$. Thus, square $(*)$ is pullback.

$(4) \Rightarrow (3)$ Let $x \in \beta_u X$ and we suppose, that $\beta_u X(x) = y \in Y$. We put $Z = \{x\}$ and determine a mapping $h : \mathrm{WZ} \to \beta_u X$, as $h(x) = x$, and $g : \mathrm{WZ} \to vY$ as $g(x) = y = \beta_u f(x) \in Y$, where W is trivial uniformity on $Z = \{x\}$. Then $\beta_u f \circ h = i_Y \circ g$. There is such *u*–continuous mapping $h' : \mathrm{WZ} \to uX$ that $h = i_X \circ h'$. So $x \in X$, i.e. $\beta_u f(\beta_u X \setminus X) \subset \beta_v Y \setminus Y$.

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