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ОЦЕНКИ СНИЗУ РЕШЕНИЙ СЛАБО НЕЛИНЕЙНОГО ВОЛЬТЕРРОВА ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА

С. Искандаров, Г.Т. Халилова

Исследуется задача установления достаточных условий для оценки снизу на полуоси и стремления к бесконечности решений слабо нелинейного ИДУ второго порядка типа Вольтерра. Показывается, что данная задача решаема для решений из определенного многообразия начальных данных.

Ключевые слова: интегро-дифференциальное уравнение типа Вольтерра; оценка снизу; стремление к бесконечности; многообразие начальных данных.

LOWER ESTIMATES OF SOLUTIONS OF WEAK NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND ORDER

S. Iskandarov, G.T. Khalilova

It is studied the problem of finding sufficient conditions for the lower estimates on the semiaxis and the tend to infinity of solutions of a weakly nonlinear Volterra integro-differential equation of second order. It is shown that this problem is solved for the solutions from certain manifold of the initial data.

Key words: Integro-differential equation of Volterra type; the lower estimate; the tend to infinity; the manifold of initial data.

All referenced functions of $t,(t,\tau)$ and their derivatives are continuous and relations occur in $t \ge t_0,\ t \ge \tau \ge t_0$; functions of x,y,z are continuous in $|x|,|y|,|z| < \infty$; $\mathcal{I} = [t_0, \infty)$; IDE stands for " integro-differential equation".

PROBLEM. Establish sufficient conditions for the lower estimates in $\mathcal{I}$ and tends to infinity under $t \to \infty$ of solutions of the second order IDE:

$$x''(t) + a_1(t)x'(t) + a_0(t)x(t) + \int_{t_0}^{t}\left[Q_0(t,\tau)x(\tau) + Q_1(t,\tau)x'(\tau)\right]d\tau = f(t) + F\left(t, x(t), x'(t), \int_{t_0}^{t} H(t,\tau,x(\tau),x'(\tau))\,d\tau\right),\ t \ge t_0, \quad (1)$$

Where $F(t,x,y,z), H(t\ \tau,x,y)$ satisfy the condition of weak nonlinearity:

$$|F(t,x,y,z)| \le g_0(t)|x| + g_1(t)|y| + g_2(t)|z|, |H(t,\tau,x,y)| \le h_0(t,\tau)|x| + h_1(t,\tau)|y|, \quad (F,H)$$

with non-negative $g_0(t), g_1(t), g_2(t), h_0(t,\tau), h_1(t,\tau)$.

This problem is studied for the first time.

We consider the solutions of IDE (1) c $\forall x^{(k)}(t_0)\ (k = 0,1)$. By virtue of conditions (F,H) such solutions exist.

Following [1], in the IDE (1) we make the following non-standard replacement:

$$x'(t) = \delta(t)x(t) + W(t)y(t), \quad (2)$$

where $\delta(t)$, $W(t)$ are some weighing function and $\delta(t) \ge 0, W(t) > 0; y(t)$ is new unknown function, and the IDE (1) reduce to the following equivalent system:

$$\begin{cases} x'(t) = \delta(t)x(t) + W(t)y(t), \\ y'(t) + b_1(t)y(t) + b_0(t)x(t) + \int_{t_0}^{t}\left[P(t,\tau)x(\tau) + \right. \\ \left. + K(t,\tau)y(\tau)\right]d\tau = q(t) + (w(t))^{-1}F(t, x(t), \delta(t)x(t) + \\ W(t)y(t), \int_{t_0}^{t} H(t,\tau,x(\tau),\delta(\tau)x(\tau) + W(\tau)y(\tau))d\tau),\ t \ge t_0, \end{cases} \quad (3)$$

where

$$b_1(t) \equiv a_1(t) + \delta(t) + W'(t)(W(t))^{-1},$$

$$b_0(t) \equiv \left[a_0(t) + \delta(t)a_1(t) + (\delta(t))^2 + \delta'(t)\right](W(t))^{-1},\ P(t,\tau) \equiv (W(t))^{-1}\left[Q_0(t,\tau) + \delta(\tau)Q_1(t,\tau)\right],$$

$$K(t,\tau) \equiv (W(t))^{-1} Q_1(t,\tau) W(\tau), q(t) \equiv f(t)(W(t))^{-1}.$$

We need an upper bound for $|y(t)|$. To do this, the system (3) apply the method of transformation equations Volterra [2, p. 194–217], the method of cutting functions [3, p. 41], the method of integral inequalities [4]. Thus, an upper bound $|y(t)|$.

Let [3]:

$$q(t) = \sum_{i=0}^{n} q_i(t), \quad \text{(q)}$$

$\psi_i(t)$ $(i = 1..n)$ – are some cutting functions,

$$R_i(t,\tau) \equiv K_i(t,\tau)(\psi_i(t)\psi_i(\tau))^{-1},$$

$$E_i(t) \equiv q_i(t)(\psi_i(t))^{-1} \ (i = 1..n)$$

$$R_i(t,t_0) = A_i(t) + B_i(t) \ (i = 1..n), \quad \text{(R)}$$

$c_i(t)(i = 1..n)$ – are some functions.

For arbitrarily fixed solution $(x(t), y(t))$ of system (3), as in [2, p. 194-217], its first equation is multiplied on the $x(t)$, the second – on the $y(t)$, we add these relations, then perform integration between t_0 to t, including parts, at the same time as in [3] we introduce conditions (K), (q), the functions $\psi_i(t), R_i(t,\tau)$, the condition (R), functions $E_i(t), c_i(t)$, using lemmas 1.4 and 1.5 [5]. Then we have the following identity:

$$u(t) \equiv (x(t))^2 + (y(t))^2 + 2\int_{t_0}^{t} b_1(s)(y(s))^2 ds +$$

$$+\sum_{i=1}^{n}\Big\{A_i(t)(Y_i(t,t_0))^2 + B_i(t)(Y_i(t,t_0))^2 -$$

$$-2E_i(t)Y_i(t,t_0) + c_i(t) - \int_{t_0}^{t}\Big[B_i'(s)(Y_i(s,t_0))^2 -$$

$$-2E_i'(s)Y_i(s,t_0) + c_i'(s)\Big]ds + \int_{t_0}^{t} R_{i\tau}'(t,\tau)(Y_i(t,\tau))^2 d\tau\Big\} \equiv c_* +$$

$$+2\int_{t_0}^{t}\delta(s)(x(s))^2 ds + 2\int_{t_0}^{t} W(s)x(s)y(s)ds +$$

$$+\sum_{i=1}^{n}\int_{t_0}^{t}\Big[A_i'(s)(Y_i(s,t_0))^2 + \int_{t_0}^{s} R_{is\tau}^{*}(s,\tau)(Y_i(s,\tau))^2 d\tau\Big]ds -$$

$$-2\int_{t_0}^{t} y(s)\Big\{q_0(s) + b_0(s)x(s) +$$

$$+\int_{t_0}^{t}\big[P(s,\tau)x(\tau) + K_0(s,\tau)y(\tau)\big]d\tau +$$

$$+(W(s))^{-1} F(s, x(s), \delta(s)x(s) +$$

$$+W(s)y(s),$$

$$\int_{t_0}^{s} H(s,\tau,x(\tau),\delta(\tau)x(\tau) + W(\tau)y(\tau))d\tau\Big\}ds,$$

$$t \ge t_0, \quad (4)$$

where

$$Y_i(t,\tau) \equiv \int_{\tau}^{t}\psi_i(\eta)y(\eta)d\eta,$$

$$c^* = (x(t_0))^2 + (y(t_0))^2 + \sum_{i=1}^{n} c_i(t_0).$$

THEOREM 1. Suppose that, the following conditions are satisfied:

1) $\delta(t) \ge 0$, $W(t) > 0$, $(K),(q),(R)$;

2) $b_1(t) \ge 0$;

3) $A_i(t) \ge 0$, $B_i(t) \ge 0$, $B_i'(t) \le 0$, $R_{i\tau}'(t,\tau) \ge 0$, there exists functions $A_i^*(t) \ge 0$, $c_i(t)$, $R_i^*(t) \ge 0$ such, that $A'_i(t) \le A_i^*(t)A_i(t), (E_i^{(k)}(t))^2 \le B_i^{(k)}(t)c_i^{(k)}(t)$,

$$R''_{its}(t,\tau) \le R_i^*(t)R'_{it}(t,\tau)(i = 1..n; K = 0,1)$$

Then for any solution $(x(t), y(t))$ of system (3) are true the estimates:

$$(x(t))^2 + (y(t))^2 \le u(t) \le$$

$$\le \left\{\sqrt{c^*} + \int_{t_0}^{t}(W(s))^{-1}|q_0(s)|E(s)e^{-\int_{t_0}^{s}\delta(\tau)d\tau}ds\right\}^2 \times (E(t))^2 e^{2\int_{t_0}^{t}\delta(s)ds}, \quad (5)$$

$$|y(t)| \le \left\{\sqrt{c^*} + \int_{t_0}^{t}(W(s))^{-1}|q_0(s)|E(s)e^{-\int_{t_0}^{s}\delta(\tau)d\tau}ds\right\}E(t)e^{\int_{t_0}^{t}\delta(s)ds}, \quad (6)$$

where

$$E(t) \equiv exp\left(\int_{t_0}^{t}\Big\{W(s) + \frac{1}{2}\sum_{i=1}^{n}\big[A_i^*(s) + R_i^*(s)\big] + |b_0(s)| +\right.$$

$$+\int_{t_0}^{s}\big[|P_0(s,\tau)| + |K_0(s,\tau)|\big]d\tau + (W(s))^{-1}$$

$$\big[g_0(s) + g_1(s)\delta(s) + g_1(s)W(s)\big] +$$

$$G_k(t,\tau) \equiv g_2(t)h_k(t,\tau) \quad (k = 0,1).$$

To prove this theorem from the identity (3) moves to the integral inequality, using the conditions (F,H), then this inequality we apply lemma 1 [4].

Next we will deal with the lower bound solutions IDE (1).

From differential equation (2) by the method of Lagrange [6, p. 391–394], we obtain the following integral representation for $x(t)$:

$$x(t)=e^{\int_{t_0}^{t}\delta(s)ds}\left[x(t_0)+\int_{t_0}^{t}e^{-\int_{t_0}^{s}\delta(\tau)d\tau}W(s)y(s)ds\right]. \quad (7)$$

From (7), as in [7,8], we have a lower bound for $|x(t)|$:

$$|x(t)|\geq e^{\int_{t_0}^{t}\delta(s)ds}\left[|x(t_0)|-\int_{t_0}^{t}e^{-\int_{t_0}^{s}\delta(\tau)d\tau}W(s)|y(s)|ds\right]. \quad (8)$$

Hence, by (6) we obtain

$$|x(t)|\geq e^{\int_{t_0}^{t}\delta(s)ds}\left[|x(t_0)|-\int_{t_0}^{t}W(s)E(s)\left\{\sqrt{c^*}+\right.\right.$$

$$\left.\left.+\int_{t_0}^{s}(W(\eta))^{-1}|q_0(\eta)|E(\eta)e^{-\int_{t_0}^{\eta}\delta(\tau)d\tau}d\eta\right\}ds\right]. \quad (9)$$

From (9) implies the following

THEOREM 2. Suppose: 1) all the conditions of theorem 1 are satisfy;

2) $M(x(t_0),x'(t_0))\equiv|x(t_0)|-\int_{t_0}^{\infty}W(s)E(s)\left\{\sqrt{c^*}+\right.$

$$\left.+\int_{t_0}^{s}(W(\eta))^{-1}|q_0(\eta)|E(\eta)e^{-\int_{t_0}^{\eta}\delta(\tau)d\tau}d\eta\right\}ds>0. \quad (10)$$

Then for any solution $x(t)$ with initial data $x(t_0)$, $x'(t_0)$ from the manifold (10) we have the following lower bound:

$$|x(t)|\geq M(x(t_0),x'(t_0))e^{\int_{t_0}^{t}\delta(s)ds}. \quad (11)$$

From theorem 2, i.e., from (11) we have

COROLLARY. If all the conditions of theorem 2 are satisfy, then the solution $x(t)$ of the IDE (1) with initial data $x^{(k)}(t_0)$ $(k=0,1)$ from (10) we have: $\lim_{t\to\infty}|x(t)|=\infty$.

NOTICE. In view of the change (2), for c^* from the manifold of initial data $M(x(t_0),x'(t_0))$ (10), we obtain:

$$c^*=(x(t_0))^2+(W(t_0))^{-2}\left[x'(t_0)-\delta(t_0)x(t_0)\right]^2+\sum_{i=1}^{n}c_i(t_0).$$

Note that a problem similar to our previously solved for the IDE (1) with $F(t,x,y,z)\equiv 0$, $\delta(t)\equiv\delta=const$ in the article the authors [9]. In this paper, we have largely adhered to the style of the material [9].

The analysis shows that for the condition (10) will play a significant role weighting function $\delta(t)$, which plays the role of the trial function and in practice it is chosen so that the whole complex of conditions of Theorems 1.2 were natural.

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