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# ЗАДАЧА С КОСОЙ ПРОИЗВОДНОЙ ДЛЯ СИСТЕМЫ УРАВНЕНИЙ НЕРАВНОВЕСНОЙ СОРБЦИИ

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Исследуется система уравнений, моделирующая процесс неравновесной сорбции. Доказывается теорема существования и единственности решения задачи с косой производной в многомерном случае в гельдеровских классах функций. Важную роль при доказательстве теоремы играет полученный принцип максимума. Существование решения задачи показывается с помощью теоремы Шаудера о неподвижной точке вполне непрерывного оператора на малом промежутке времени. Затем получены оценки, позволяющие продолжить решение до любого конечного значения времени.

*Ключевые слова:* процесс неравновесной сорбции; задача с косой производной; глобальная однозначная разрешимость.

# ТЕҢ САЛМАКСЫЗ СОРБЦИЯНЫН ТЕҢДЕМЕЛЕР СИСТЕМАСЫ ҮЧҮН ЖАНТЫК ТУУНДУ МЕНЕН МАСЕЛЕ

Бул макалада тең салмаксыз сорбция процессин моделдөөчү теңдемелер системасы изилдөөгө алынган. Гельдердик функциялар классында көп ченемдүү учурда жантык туунду менен маселени чечүүнүн жападан жалгыз жолу бар экендиги тууралуу теорема далилденет. Теореманы далилдөөдө алынган максимум принциби маанилүү роль ойнойт. Аз убакыт аралыгында тынымсыз кыймылдагы оператордун кыймылсыз чекити тууралуу Шаудердин теоремасынын жардамы менен маселени чечүүнүн жолу бар экендиги көрсөтүлөт. Андан соң кайсы гана убакыт маанисине чейин болбосун маселени чечүүнү улантууга мүмкүндүк берүүчү баалар алынды.

*Түйүндүү сөздөр:* тең салмаксыз сорбция процесси; жантык туунду менен маселе; глобалдуу бир мааниде чечилүү мүмкүндүгү.

# OBLIQUE DERIVATIVE PROBLEM FOR THE SYSTEM OF EQUATION OF NON-EQUILIBRIUM SORPTION

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The article regards a system of equations that simulates the process of non-equilibrium sorption. The existence and uniqueness theorem for the solution of the oblique derivative problem in the multidimensional case in the holder classes of functions is proved. An important role in the proof of the theorem is played by the maximum principle obtained. The existence of a solution of the problem is shown using the Schauder theorem on the fixed point of a completely continuous operator on a small time interval. Then estimates are obtained that allow the solution to continue to any finite time value.

*Keywords:* process of non-equilibrium sorption; oblique derivative problem; global single-valued solvability.

***Introduction.*** Actually all liquids found in nature are solutions, i.e. a mixture of two or more substances (components). The filtration of liquids and gases associated with them (dissolved, suspended) solids in porous media accompanies the diffusion of these substances and the mass exchange between the liquid (gas) and solid phases. The most common types of mass transfer are sorption and desorption, ion exchange, dissolution and crystallization, colmatation, sulfation and suffusion, paraffinization. Taking into account the physical and chemical interaction of the solutions with the formation rocks, the problems of equilibrium and non-equilibrium sorption are considered.

In this paper, we prove a global unique solvability of the oblique derivative problem that simulates the process of non-equilibrium sorption.

***Formulation of the problem.*** Let $m(x,t)$ is the porosity of the medium, $0 < m(x,t) \le 1$; pore space is filled with the solution and solid phase precipitated from the solution; $c(x,t)$ is a mass concentration of a certain substance in liquid phase (per unit volume of solution); $s(x,t)$ is a mass concentration of the solid phase of the substance the precipitated (per unit pore volume).

Under equilibrium conditions, when the contact between the solution and the solid phase is maintained for a long enough time, the ratio between the concentrations $c(x,t)$ in solution and $s(x,t)$ on the sorbent is determined by sorption isotherm. At low concentrations of the solution, the amount of absorption is determined by the linear relationship Henry isotherm $s = \Gamma c$, where $\Gamma > 0$ is a certain constant depending on the physical and chemical properties of the medium (the Henry constant).

Equilibrium sorption equations can not always fully characterize the features of absorption and metabolism in a two-phase solution – solid phase system. In works [1–3] were proposed mathematical models for describing the processes of non-equilibrium sorption. The concentration of the solid phase $s(x,t)$ is associated with the concentration $c(x,t)$ in the liquid phase with the equation

$$\frac{\partial s}{\partial t} = \frac{1}{\tau}\left(\Gamma c - s\right), \tag{1}$$

where the positive constant $\tau$ is the characteristic relaxation time, $\Gamma$ is the Henry's constant. The concentration $c$ of the substance in solution satisfies the equation

$$m\frac{\partial c}{\partial t} = D\Delta c - \mathbf{v}\cdot\nabla c - \frac{\partial s}{\partial t}, \tag{2}$$

where $D(x,t) > 0$ is the diffusion coefficient, $\mathbf{v}(x,t)$ is the vector of the filtration rate, which are considered known functions of these arguments; $\Delta$ is the Laplace operator, $\nabla$ is the gradient, $\mathbf{v}\cdot\nabla c$ denotes the scalar product of the vectors $\mathbf{v}$ and $\nabla c$.

In [4] the global unique solvability of the first initial-boundary value problem for system (1) – (2) is proved. In [5-7], a difference approximation of the differential problem was formulated using an implicit scheme, a solution of the difference problem was constructed using the sweep method, and the results of numerical experiments were presented.

In the present paper we consider the oblique derivative problem for the system of equations (1)–(2), describing the process of non-equilibrium sorption.

Let $\Omega$ is a bounded domain of n-dimensional space $R^n$ with a sufficiently smooth boundary $S = \partial\Omega$, $Q_T = \Omega\times(0,T)$, $T > 0$; $S_T = S\times(0,T)$ is cylinder lateral surface of $Q_T$. It is required to find the functions $c(x,t)$, $s(x,t)$, defined in domain $Q_T$ satisfying in $Q_T$ the equations (1), (2), when the initial conditions

$$c(x,0) = c_0(x),\ x\in\Omega, \tag{3}$$

$$s(x,0) = s_0(x),\ x\in\Omega, \tag{4}$$

and the boundary condition with oblique derivative are fulfilled:

$$\sum_{i=1}^{n} b_i(x,t)\frac{\partial c(x,\,t)}{\partial x_i} = 0,\ (x,\,t)\in S_T. \tag{5}$$

Suppose that a vector field

$\mathbf{b}(x,t) = (b_1(x,t), b_2(x,t), \dots, b_n(x,t))$

does not lie in the tangent plane at $S$ and $\mathbf{b}\cdot\mathbf{n} \le -\varepsilon < 0$, where $\mathbf{n}(\xi)$ is the unit vector of the outward normal to $S$ at the point $\xi$.

The main result of the paper is the following

***Theorem.*** Let $0 < \alpha < 1$ is a certain number, the boundary $S$ of the domain belongs to the Hölder class $C^{2+\alpha}$, the coefficients $m, D, \mathbf{v}$ of the equation (2) belong to the Holder class

$$C^{\alpha,\alpha/2}\ (\bar{Q}_T),$$

functions

$$c_0(x)\in C^{2+\alpha}\ (\bar{\Omega}),$$
$$s_0(x)\in C^{\alpha}\ (\bar{\Omega}),$$
$$\mathbf{b}(x,t)\in C^{1+\alpha,(1+\alpha)/2}\ (\bar{S}_T),$$

the compatibility conditions of the zero order are satisfied:

$$\sum_{i=1}^{n} b_i(x,0)\frac{\partial c_0(x)}{\partial x_i} = 0,\ x\in S$$

and the conditions

$$0 \le c_0(x) \le M,\ 0 \le s_0(x) \le \Gamma M,\ x\in\Omega.$$

are fulfilled. Then the problem (1)–(5) has a unique classical solution

$$c(x,t)\in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T),$$

$$s(x,t)\in C^{\alpha,1+\alpha/2}(\bar{Q}_T)$$

and estimates

$$0 \le c(x,t) \le M,$$
$$0 \le s(x,t) \le \Gamma M,\ (x,t)\in Q_T$$

are valid.

***Proof of the theorem.***

First, estimates are obtained that represent the maximum principle

$$0 \le c(x,t) \le M,\ (x,t) \in Q_T, \quad (6)$$

$$0 \le s(x,t) \le \Gamma M,\ (x,t) \in Q_T. \quad (7)$$

From (1) and (3) we obtain the representation

$$s(x,t) = s_0(x)e^{-t/\tau} + \frac{\Gamma}{\tau} e^{-t/\tau} \int_0^t c(x,\theta)e^{\theta/\tau} d\theta. \quad (8)$$

Substituting (8) into (2), we obtain

$$m\frac{\partial c}{\partial t} - D\Delta c + \mathbf{i} \cdot \nabla c + \frac{\Gamma}{\tau} c = \frac{1}{\tau} s_0(x)e^{-t/\tau} + \frac{\Gamma}{\tau^2} e^{-t/\tau} \int_0^t c(x,\theta)e^{\theta/\tau} d\theta. \quad (9)$$

Suppose that the negative minimum $c_{\min} < 0$ of the function $c(x,t)$ is attained at some point $(x_0,t_0)$ inside the domain $Q_T$.. Then at this point $c_t \le 0,\ -\Delta c \le 0,\ \nabla c = 0$ and from (9) we obtain

$$\frac{\Gamma}{\tau} c_{\min} \ge \frac{1}{\tau} s_0(x_0)e^{-t_0/\tau} + \frac{\Gamma}{\tau^2} c_{\min} e^{-t_0/\tau} \int_0^{t_0} e^{\theta/\tau} d\theta,$$

$$\Gamma c_{\min} \ge s_0(x_0)e^{-t_0/\tau} + \Gamma c_{\min} e^{-t_0/\tau}\left(e^{t_0/\tau} - 1\right),$$

$$0 \ge s_0(x_0)e^{-t_0/\tau} - \Gamma c_{\min} e^{-t_0/\tau},$$

that is, they got a contradiction, because $s_0(x) \ge 0$ and $c_{\min} < 0$. Consequently, the negative minimum of the function $c(x,t)$ can not be achieved within the region $Q_T$.

On the boundary $S_T$ the minimum can not be achieved by condition (5) and the Zaremba – Giraud lemma.

***Lemma.*** Let

$$Lu = \sum_{i,j=1}^{n} a_{i,j}(x)u_{x_i x_j}(x) + \sum_{i=1}^{n} b_i(x)u_{x_i}(x)$$

an elliptic operator in a bounded domain $\Omega$, with a sufficiently smooth boundary $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $Lu \le 0$ in $\Omega$ and let the function $u(x)$ reach a strict global minimum at the boundary point $x_0 \in \partial\Omega$. Then for any vector $\mathbf{b}$ satisfying the condition $\mathbf{b} \cdot \mathbf{n} < 0$ the inequality

$$\left.\frac{\partial u}{\partial \mathbf{b}}\right|_{x_0} > 0$$

is satisfied, where $\mathbf{n}$ is the outer normal to $\partial\Omega$ at the point $x_0$.

This lemma for harmonic functions was proved by Zaremba [8], and in a more general formulation of Giraud [9].

In our case, we consider:

$$Lc = D\Delta c - \mathbf{i} \cdot \nabla c = F(x,t) = \frac{\partial c}{\partial t} + \frac{\Gamma}{\tau} c - \frac{1}{\tau} s_0(x)e^{-t/\tau} - \frac{\Gamma}{\tau^2} e^{-t/\tau} \int_0^t c(x,\theta)e^{\theta/\tau} d\theta.$$

Suppose that the negative minimum $c_{\min} < 0$ of the function $c(x,t)$ is attained at some point $(x_0,t_0)$ on the boundary of the region $S_T$. Then:

$$F(x_0,t_0) = \frac{\partial c}{\partial t}(x_0,t_0) + \frac{\Gamma}{\tau} c_{\min} - \frac{1}{\tau} s_0(x_0)e^{-t_0/\tau} - \frac{\Gamma}{\tau^2} e^{-t_0/\tau} \int_0^{t_0} c(x_0,\theta)e^{\theta/\tau} d\theta \le$$

$$\le \frac{\partial c}{\partial t}(x_0,t_0) + \frac{\Gamma}{\tau} c_{\min} - \frac{1}{\tau} s_0(x_0)e^{-t_0/\tau} - \frac{\Gamma}{\tau^2} c_{\min} e^{-t_0/\tau} \int_0^{t_0} e^{\theta/\tau} d\theta = \frac{\partial c}{\partial t}(x_0,t_0) + \frac{\Gamma}{\tau} c_{\min} - \frac{1}{\tau} s_0(x_0)e^{-t_0/\tau} - \frac{\Gamma}{\tau} c_{\min} e^{-t_0/\tau}(e^{t_0/\tau} - 1) =$$

$$= \frac{\partial c}{\partial t}(x_0,t_0) - \frac{1}{\tau} s_0(x_0)e^{-t_0/\tau} + \frac{\Gamma}{\tau} c_{\min} e^{-t_0/\tau} < 0.$$

Hence $F(x,t_0) < 0$ in a neighborhood of a point $x_0$ and one can apply the Zaremba – Giraud lemma, i. e.

$$\left.\frac{\partial \tilde{n}}{\partial \mathbf{b}}\right|_{x_0,t_0} = \sum_{i=1}^{n} b_i(x_0,t_0)\frac{\partial c(x_0,\ t_0)}{\partial x_i} > 0.$$

But this contradicts the boundary condition (5).

Thus, the minimum of the function $c(x,t)$ is achieved at the lower boundary of the region $Q_T$, i. e. at the initial time. At the initial time, the function $c_0(x)$ is nonnegative. Thus, we have proved that $c(x,t) \ge 0,\ (x,t) \in Q_T$.

Suppose now that within the region $Q_T$ a positive maximum $c_{\max} > M$ of the function $c(x,t)$ is attained, i. e. there exists a point $(x_1,t_1) \in Q_T : c(x_1,t_1) = c_{\max} > M$. At this point $c_t \ge 0,\ -\Delta c \ge 0,\ \nabla c = 0,$ and from (9) we obtain the inequalities:

$$\frac{\Gamma}{\tau} c_{\max} \le \frac{1}{\tau} s_0(x_1)e^{-t_1/\tau} + \frac{\Gamma}{\tau^2} c_{\max} e^{-t_1/\tau} \int_0^{t_1} e^{\theta/\tau} d\theta,$$

$$\Gamma c_{\max} \le s_0(x_1)e^{-t_1/\tau} + \Gamma c_{\max} e^{-t_1/\tau}\left(e^{t_1/\tau} - 1\right),$$

$$0 \le s_0(x_1)e^{-t_1/\tau} - \Gamma c_{\max} e^{-t_1/\tau} = \left(s_0(x_1) - \Gamma c_{\max}\right)e^{-t_1/\tau}.$$

Again we have a contradiction, because $s_0(x) \le \Gamma M$, and $c_{\max} > M$.

The maximum of the function $c(x,t)$ can not be reached on the boundary $S_T$ because of condition (5) and the Zaremba – Giraud lemma. Let

$$Lc = D\Delta c - \mathbf{í}\cdot\nabla c = F(x,t) =$$
$$= \frac{\partial c}{\partial t} + \frac{\Gamma}{\tau}c - \frac{1}{\tau}s_0(x)e^{-t/\tau} - \frac{\Gamma}{\tau^2}e^{-t/\tau}\int_0^t c(x,\theta)e^{\theta/\tau}d\theta.$$

Suppose that the positive maximum $c_{\max} > M$ of the function $c(x,t)$ is reached at some point $(x_1, t_1)$ on the boundary of the domain $S_T$. Then:

$$F(x_1,t_1) = \frac{\partial c}{\partial t}(x_1,t_1) + \frac{\Gamma}{\tau}c_{\max} - \frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} -$$
$$- \frac{\Gamma}{\tau^2}e^{-t_1/\tau}\int_0^{t_1} c(x_1,\theta)e^{\theta/\tau}d\theta \geq$$
$$\geq \frac{\Gamma}{\tau}c_{\max} - \frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} - \frac{\Gamma}{\tau^2}c_{\max}e^{-t_1/\tau}\int_0^{t_1} e^{\theta/\tau}d\theta =$$
$$= \frac{\Gamma}{\tau}c_{\max} - \frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} - \frac{\Gamma}{\tau}c_{\max}e^{-t_1/\tau}\left(e^{t_1/\tau} - 1\right) =$$
$$= -\frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} + \frac{\Gamma}{\tau}c_{\max}e^{-t_1/\tau} =$$
$$= \frac{1}{\tau}\left(\Gamma c_{\max} - s_0(x_1)\right)e^{-t_1/\tau} > \frac{1}{\tau}\left(\Gamma M - \Gamma M\right)e^{-t_1/\tau} = 0.$$

Consequently, $F(x,t_1) > 0$ in the neighborhood of the point $x_1$ and one can apply the Zarembo–Giraud lemma, i. e.

$$\left.\frac{\partial c}{\partial \mathbf{b}}\right|_{x_1,t_1} < 0 .$$

But this contradicts the boundary condition (5).

Thus, the maximum of the function $c(x,t)$ is achieved at the lower boundary of the region $Q_T$, i. e. at the initial time. At the initial instant of time, the function $c_0(x) \leq M$. Therefore, $c(x,t) \leq M$, $(x,t) \in Q_T$. The estimate (6) is proved.

The estimate (7) follows from the representation (8) using (6). In fact, since $s_0(x) \geq 0$, $c(x,t) \geq 0$, it follows from (8) that $s(x,t) \geq 0$, $(x,t) \in Q_T$.

Since $s_0(x) \leq \Gamma M$, $c(x,t) \leq M$, then

$$s(x,t) \leq s_0(x)e^{-t/\tau} + \frac{\Gamma M}{\tau}e^{-t/\tau}\int_0^t e^{\theta/\tau}d\theta \leq \Gamma M e^{-t/\tau} +$$
$$+ \Gamma M e^{-t/\tau}\left(e^{t/\tau} - 1\right) = \Gamma M.$$

The estimate (7) is proved.

The uniqueness of the solution of problem (1)–(5) is a consequence of the estimates (6)–(7).

The existence of a solution of problem (1)–(5) is proved with the help of Schauder's theorem on the fixed point of a completely continuous operator. Denote by $V_{T_1}$ the next closed convex subset of $C^{2+\alpha,1+\alpha/2}(\bar{Q}_{T_1})$:

$$V_{T_1} = \left\{ \begin{matrix} \tilde{c}(x,t)\left|\tilde{c}(x,0) = c_0(x);\ \dfrac{\partial \tilde{c}(x,t)}{\partial \mathbf{b}} = 0,\right. \\ (x,t) \in S_{T_1};\ \|\tilde{c}\|_{\tilde{N}^{2+\alpha,1+\alpha/2}(\bar{Q}_{T_1})} \leq K \end{matrix} \right\},$$

where $K$ is some fixed positive number depending on the data of problem (1)–(5), which we will define later. By a given function $\tilde{c} \in V_{T_1}$ we find the function

$$\tilde{s}(x,t) = s_0(x)e^{-t/\tau} + \frac{\Gamma}{\tau}e^{-t/\tau}\int_0^t \tilde{c}(x,\theta)e^{\theta/\tau}d\theta. \qquad (10)$$

Now to each function $\tilde{c} \in V_{T_1}$ we put the function $c = \Lambda(\tilde{c})$ as a solution of the problem

$$m\frac{\partial c}{\partial t} - D\Delta c + \mathbf{í}\cdot\nabla c + \frac{\Gamma}{\tau}c = \frac{1}{\tau}\tilde{s}, \qquad (11)$$

$$c(x,0) = c_0(x),$$
$$x \in \Omega;\ \sum_{i=1}^{n} b_i(x,t)\frac{\partial c(x,t)}{\partial x_i} = 0,\ (x,t) \in S_{T_1}. \qquad (12)$$

Let us prove that the operator $\Lambda$ is completely continuous and, for sufficiently small $T_1$, takes the set $V_{T_1}$ into itself.

Let us show that $\tilde{s} \in C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)$. It follows from (10) that

$$|\tilde{s}|^{(0)}_{Q_{T_1}} \equiv \max_{(x,t)\in\bar{Q}_{T_1}} |\tilde{s}(x,t)| \leq |s_0|^{(0)}_{\Omega} + \Gamma|\tilde{c}|^{(0)}_{Q_{T_1}} \max_{t\in[0,T_1]}\left(1 - \mathrm{e}^{-t/\tau}\right).$$

Hence, using the expansion of the function $\mathrm{e}^{-t/\tau}$ in the Maclaurin series, it is easy to obtain (for $T_1 < \tau$) the estimate

$$|\tilde{s}|^{(0)}_{Q_{T_1}} \leq |s_0|^{(0)}_{\Omega} + T_1\frac{\Gamma}{\tau}|\tilde{c}|^{(0)}_{Q_{T_1}}. \qquad (13)$$

Similarly, from (10) follows the estimate:

$$|\tilde{s}|^{(\alpha)}_{x,Q_{T_1}} \equiv \sup_{(x,t),(x',t)\in\bar{Q}_{T_1}} \frac{|\tilde{s}(x,t) - \tilde{s}(x',t)|}{|x - x'|^{\alpha}} \leq$$
$$\leq |s_0|^{(\alpha)}_{x,\Omega} + \Gamma|\tilde{c}|^{(\alpha)}_{x,Q_{T_1}} \max_{t\in[0,T_1]}\left(1 - \mathrm{e}^{-t/\tau}\right) \leq$$
$$\leq |s_0|^{(\alpha)}_{x,\Omega} + T_1\frac{\Gamma}{\tau}|\tilde{c}|^{(\alpha)}_{x,Q_{T_1}}, \qquad (14)$$

i. e. the function $\tilde{s}$ satisfies the Holder condition with respect to the space variable with exponent α

The function $\tilde{s}$ satisfies the Holder condition with respect to the variable $t$ with any exponent $0 < \beta \leq 1$ (even Lipschitz), since it has a bounded derivative with respect to time

$$\tilde{s}_t(x,t) = -\frac{1}{\tau}s_0(x)e^{-t/\tau} - \frac{\Gamma}{\tau^2}e^{-t/\tau}\int_0^t \tilde{c}(x,\theta)e^{\theta/\tau}d\theta + \frac{\Gamma}{\tau}\tilde{c}(x,t),$$

$$|\tilde{s}_t|^{(0)}_{Q_{T_1}} \leq \frac{1}{\tau}|s_0|^{(0)}_{\Omega} + \frac{\Gamma}{\tau}|\tilde{c}|^{(0)}_{Q_{T_1}} \max_{t\in[0,T_1]}\left(1 - \mathrm{e}^{-t/\tau}\right) +$$
$$+ \frac{\Gamma}{\tau}|\tilde{c}|^{(0)}_{Q_{T_1}} \leq \frac{1}{\tau}|s_0|^{(0)}_{\Omega} + \frac{2\Gamma}{\tau}|\tilde{c}|^{(0)}_{Q_{T_1}}. \qquad (15)$$

In particular, with $\beta = 1$ we have:

$$\frac{|\tilde{s}(x,t) - \tilde{s}(x,t^{/})|}{|t - t^{/}|^{\alpha/2} |t - t^{/}|^{1-\alpha/2}} \le |\tilde{s}_t|^{(0)}_{Q_{T_1}} .$$

This implies the inequality:

$$|\tilde{s}|^{(\alpha/2)}_{t,Q_{T_1}} \equiv \sup_{(x,t),(x,t^{/}) \in \bar{Q}_{T_1}} \frac{|\tilde{s}(x,t) - \tilde{s}(x,t^{/})|}{|t - t^{/}|^{\alpha/2}} \le T_1^{1-\alpha/2} |\tilde{s}_t|^{(0)}_{Q_{T_1}} . \quad (16)$$

Estimates (13)–(16) prove that $\tilde{s} \in C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)$ and under the condition $T_1 < 1$ the next estimate holds

$$\|\tilde{s}\|_{\tilde{N}^{\alpha,\alpha/2}(\bar{Q}_{T_1})} \le C_1 \|s_0\|_{\tilde{N}^{\alpha}(\Omega)} + T_1^{1-\alpha/2} C_2 \|\tilde{c}\|_{\tilde{N}^{\alpha,\alpha/2}(\bar{Q}_{T_1})}, \quad (17)$$

where $C_1, C_2$ are some positive constants that do not depend on $s_0, \tilde{c}$. We will assume that $C_1, C_2$ depends on $T$, but does not depend on $T_1 < \min\{T, 1, \tau\}$.

Since

$$\tilde{s}_t = \frac{1}{\tau}(\Gamma \tilde{c} - \tilde{s}),$$

then $\tilde{s} \in C^{\alpha,1+\alpha/2}\left(\bar{Q}_{T_1}\right)$.

For a solution $c(x,t)$ of the problem (11), (12), the estimate [10, p. 365] is valid

$$\|c\|_{\tilde{N}^{2+\alpha,1+\alpha/2}(\bar{Q}_{T_1})} \le C\left(\|c_0\|_{\tilde{N}^{2+\alpha}(\bar{\Omega})} + \|\tilde{s}\|_{\tilde{N}^{\alpha,\alpha/2}(\bar{Q}_{T_1})}\right), \quad (18)$$

where $C$ is a positive constant independent of $c_0, \tilde{s}$. We will assume that $C$ depends on $T$, but does not depend on $T_1 < T$. Using (17), (18), we have

$$\|c\|_{C^{2+\alpha,1+\alpha/2}(\bar{Q}_{T_1})} \le C_3\left(\|c_0\|_{C^{2+\alpha}(\bar{\Omega})} + \|s_0\|_{C^{\alpha}(\bar{\Omega})}\right) +$$
$$+ T_1^{1-\alpha/2} C_4 \|\tilde{c}\|_{C^{\alpha,\alpha/2}(\bar{Q}_{T_1})} . \quad (19)$$

This implies that the operator $\Lambda : \tilde{c} \to c$ is completely continuous.

We choose the constant $K$, that appears in the definition of the set $V_{T_1}$, as outcome of the condition

$$K > C_3\left(\|c_0\|_{C^{2+\alpha}(\bar{\Omega})} + \|s_0\|_{C^{\alpha}(\bar{\Omega})}\right).$$

For definiteness, we set

$$K = 2C_3\left(\|c_0\|_{C^{2+\alpha}(\bar{\Omega})} + \|s_0\|_{C^{\alpha}(\bar{\Omega})}\right).$$

Then it follows from (19) that for sufficiently small $T_1$ the operator $\Lambda$ takes the set $V_{T_1}$ into itself.

By Schauder's theorem on the fixed point of a completely continuous operator, the set $V_{T_1}$ contains a fixed point $\tilde{c}$, which together with its corresponding function $\tilde{s}$ from (10) is a solution of problem (1)–(5) on the time interval $[0, T_1]$.

The solution can be continued in $k$ steps to $[T_k, T_{k+1}]$, $k = 1, 2, \ldots$, and $T_{k+1} - T_k \ge \delta > 0$ and $\delta$ does not depend on the number $k$. This can be seen from the estimate (19)

$$C_3\left(\|c_0\|_{C^{2+\alpha}(\bar{\Omega})} + \|s_0\|_{C^{\alpha}(\bar{\Omega})}\right) +$$
$$+ T_1^{1-\alpha/2} C_4 \|\tilde{c}\|_{C^{\alpha,\alpha/2}(\bar{Q}_{T_1})} < K =$$
$$= 2C_3\left(\|c_0\|_{C^{2+\alpha}(\bar{\Omega})} + \|s_0\|_{C^{\alpha}(\bar{\Omega})}\right).$$

Because $\|\tilde{c}\|_{\tilde{N}^{\alpha,\alpha/2}(\bar{Q}_{T_1})} \le K$, then it follows that as $\delta$ can be chosen

$$\delta^{1-\alpha/2} = \frac{\hat{E}}{2C_4 K} = \frac{1}{2C_4},$$

not depending on the number $k$. Thus, the solution in a finite number of steps can be continued to any $0 < T < +\infty$.

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