УДК 515.12

О β-ПОДОБНОЙ КОМПАКТИФИКАЦИИ РАВНОМЕРНЫХ ПРОСТРАНСТВ

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Получена характеристика базы coz-тонких равномерных пространств и для β-подобной компактификации доказан равномерный аналог теоремы Э. Чеха.

Ключевые слова: u-открытые, u-замкнутые множества; coz-отображение; нормальная база; coz-тонкое пространство.

ON β-LIKE COMPACTIFICATION OF THE UNIFORM SPACES

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The characterization of bases of coz-fine uniform spaces have been obtained and for β-like compactification the uniform analogue of E. Chekh theorem have been proved.

Keywords: u-open, u-closed sets; coz-mapping; normal base; coz-fine space.

1. Introduction. Denotations and basic properties of a uniform spaces and compactifications from [19], [2], [23]. We denote by $U(uX)$ ($U^*(uX)$) the set of all (bounded) uniformly continuous functions on the uniform space uX. The natural uniformity on uX, generated by $U(uX)$ ($U^*(uX)$), be $u_c(u_p)$ is the smallest uniformity on X with respect to its all functions from $U(uX)$ ($U^*(uX)$) are uniformly continuous. Evidently, $u_p \subseteq u_c \subseteq u_\omega \subseteq u$, where a base of uniformity u_ω is formed by all countable coverings of u. *Samuel compactification* s_uX is a completion of X with respect to uniformity u_p. Z_u is a ring of zero-sets of functions from $U(uX)$ or $U^*(uX)$ and CZ_u is a ring of cozero-sets of functions from $U(uX)$ or $U^*(uX)$. CZ_u consists of complements of sets of Z_u and, vice versa. We note, that all sets of $CZ_u\,(Z_u)$ coincide with set of all *u – open* (*u – closed*) sets in sense M. Charalambous [3], [4] of the uniform space uX. Z_u forms *separating, nest-generated intersection ring* on X and hence it is a *normal base* [9], [21], [13].

We denote the set of all natural numbers by $\mathbb{N}$, $\mathbb{R}$ is the real line, uniformity $u_{\mathbb{R}}$ on $\mathbb{R}$, is induced by the ordinary metric; for $X \subset Y$ $[X]_Y$ be a closure X in Y, for a compactum X we always use its a unique uniformity.

For *fine* uniformity u_f on Tychonoff space X [8], [19] every continuous function is uniformly continuous, hence $U(u_fX) = C(X)$ ($U^*(u_fX) = C^*(X)$) is the set of all (bounded) continuous functions on X and $Z_{u_f} = Z(X)$ is the set of all zero-sets, $CZ_{u_f} = CZ(X)$ is the set of all cozero-sets on X [8], [14]. Every maximal z_u – filter on Z_u is denoted as z_u – *ultrafilter* and z_{u_f} – *ultrafilter* on $Z(X)$ is denoted as z – *ultrafilter* [14].

A covering of z – open sets is called u – *open* and a covering of cozero-sets is called *cozero-set covering*.

Definition 1.1. A mapping $f : uX \to vY$ is called *coz – mapping*, if $f^{-1}(CZ_v) \subseteq CZ_u$ (or $f^{-1}(Z_v) \subseteq Z_u$) [10], [11]. A mapping $f : uX \to Y$ of the uniform space uX into Tychonoff space Y is called z_u – *continuous*, if $f^{-1}(CZ(X)) \subseteq CZ_u$ (or $f^{-1}(Z(Y)) \subseteq Z_u$) [7].

Evidently, that every uniformly continuous mapping is a *coz* – mapping and converse, generally speaking, incorrectly [4], [5]. Also, every z_u – continuous mapping $f : uX \to Y$ is *coz* – mapping of $f : uX \to vY$ for any uniformity v on Y. If Y is a Lindelöf or (Y, ρ) is a metric space, then its *coz* – mapping is a z_u – continuous (see. for example, [4], [5]). If $Y = \mathbb{R}$ is a real number set, or $Y = I = [0,1]$ is a unit interval *coz* – mapping of $f : uX \to \mathbb{R}$ is called *u – continuous function* and *coz* – mapping of $f : uX \to I$ is called *u – function* [4], [5].

We denote as $C_u(X)$ ($C_u^*(X)$) the set of all (bounded) u – continuous functions on the uniform space uX and $Z(uX)$ be a ring of zero-sets functions from $C_u(X)$ or ($C_u^*(X)$) and $CZ(uX)$ consists of complements of sets of $Z(uX)$ and, vice versa.

We formulate some statements without proofs, proved by A. A. Chekeev in his paper "Uniformities for Wallman compactifications and realcompactifications", it is submitted and will be published in journal "Topology and its Applications".

Proposition 1.2. *On uniform spaces uX the set* $\mathcal{B}_p^*$ $(\mathcal{B}_\omega^*)$ *of all finite (countable) u – open coverings is the base of uniformity* u_p^z (u_ω^z). *Moreover* $u_p \subseteq u_p^z$, $u_p \subseteq u_c \subseteq u_\omega \subseteq u_\omega^z$.

Proposition 1.3. *$C_u(X)$ forms a complete subring of $C_u(X)$ with inversion. It contains constant functions, separates points and closed sets, is uniformly closed and is closed under inversion, i.e. if* $f \in C_u(X)$ *and* $f(x) \neq 0$ *for all* $x \in X$ *then* $1/f \in C_u(X)$ (and an algebra in sense of [15], [16], [18]).

Lemma 1.4.

(1) *coz – mapping f : uX → vY into a compact space vY is uniformly continuous mappings* $f: u_p^z X \to vY$;

(1') *coz – mapping f : uX → vY into* $\aleph_0$*– bounded uniform space vY is uniformly continuous mappings* $f: u_\omega^z X \to vY$.

(2) $U(uX) = U(u_c X) = U(u_\omega X) \subset U(u_\omega^z X) = C_u(X)$;

(2') $U(u_p X) = U^*(uX) \subset U(u_p^z X) = U^*(u_\omega^z X) = C_u^*(X) \subset C_u(X)$.

(3) $\mathcal{Z}_u = \mathcal{Z}_{u_p} = \mathcal{Z}_{u_c} = \mathcal{Z}_{u_\omega} = \mathcal{Z}_{u_p^z} = \mathcal{Z}_{u_\omega^z} = \mathcal{Z}(uX)$.

(4) *$!_u(X)$ is a complete ring of functions with inversion on X.*

Corollary 1.5. *(1)* u_p^z *is the smallest uniformity on X with respect to which coz – mapping into a compactum vY is uniformly continuous.*

(2) u_ω^z *is the smallest uniformity on X with respect to which every coz – mapping f : uX → vY into an* $\aleph_0$ *– bounded uniform space vY is uniformly continuous.*

Let $\omega(X, \mathcal{Z}_u)$ be a Wallman compactification of with respect to the normal base $\mathcal{Z}_u$ [9]. We note that $\omega(X, \mathcal{Z}_u)$ is β – like compactification of *X* [20] and put $\beta_u X = \omega(X, \mathcal{Z}_u)$.

Theorem 1.6. *For a uniform space uX the following compactifications of X coincide:*

(1) The completion of *X* with respect to u_p^z.

(2) *The Wallman compactification* $\omega(X, \mathcal{Z}_u)$ *of X with respect to the normal base* $\mathcal{Z}_u$.

(3) *The compactification which is the set of all maximal ideals of* $C_u^*(X)$ *equipped with Stone topology* [22].

Corollary 1.7. *Every coz – mapping f : uX → vY can be extended to the continuous mapping* $\beta f : \beta_u X \to \beta_v Y$. *The first axiom of countability doesn't hold in any point* $x \in \beta_u X \setminus X$. *For uniform spaces uX and u'X we have* $\beta_u X = \beta_{u'} X$ *if and only if* $\mathcal{Z}_u = \mathcal{Z}_{u'}$.

Theorem 1.8. *For a uniform space uX the following conditions are equivalent:*

(1) Samuel compactification $s_u X$ of *uX* is a β – like compactification of *X*;

(2) $u_p = u_p^z$;

(3) any *coz* – mapping *f : uX → K* into a compactum *K* can be extended to $s_u X$;

(4) any *u* – function *f : uX → I* into a *I* can be extended to $s_u X$;

(5) if $Z_1, Z_2 \in \mathcal{Z}_u$ and $Z_1 \cap Z_2 = \varnothing$ then $[Z_1]_{s_u X} \cap [Z_2]_{s_u X} = \varnothing$;

(6) $[Z_1]_{s_u X} \cap [Z_2]_{s_u X} = [Z_1 \cap Z_2]_{s_u X}$ holds for any $Z_1, Z_2 \in \mathcal{Z}_u$;

(7) every point of $s_u X$ is the limit point for a unique z_u – ultrafilter on *uX*;

(8) every z_u – ultrafilter is a Cauchy filter with respect to u_p.

2. On compactifications of *coz*-fine uniform spaces.

Definition 2.1. [10] A uniform space *uX* is called *Alexandroff space* if its each finite *u* – open covering is uniform.

Theorem 2.2. *For a uniform space uX* $s_u X = \beta_u X$ *if and only if uX is an Alexandroff space.*

Proof. Let $s_u X = \beta_u X$ for a uniform space *uX*. Then $u_p = u_p^z$, i.e. finite *u* – open covering is uniform, hence *uX* is an Alexandroff space.

Conversely, if a uniform space *uX* is an Alexandroff space then $u_p = u_p^z$, hence $s_u X = \beta_u X$ (see Theorem 1.8). Q.E.D.

Definition 2.3 [11] A mapping $f : uX \to vY$ is called a *coz – homeomorphism,* if f maps X onto Y in a one-to-one way, and the inverse mapping $f^{-1} : vY \to uX$ is *coz* – mapping. A two uniform spaces uX and vY are *coz – homeomorphic* to each other if there exists a *coz* – homeomorphism of uX onto vY.

The next theorem is a uniform analogue of E. Čech Theorem [6].

Theorem 2.4. *Let uX and vY be the first-countable uniform spaces. Then uX is coz – homeomorphic to vY if and only if $\beta_u X$ is homeomorphic to $\beta_v Y$.*

Proof. If uX is *coz* – homeomorphic to vY, then evidently, that $\beta_u X$ is homeomorphic to $\beta_v Y$ (Corollary 1.7).

Conversely, if $\beta_u X$ is homeomorphic to $\beta_v Y$, then the uniform spaces $u_p^z X$ and $v_p^z Y$ are uniformly homeomorphic to each other (items 1, 2 of Theorem 1.8) and all points with the first–countability axiom of $\beta_u X$ transferring to all points with the first–countability axiom of $\beta_v Y$, i.e. X is *coz* – homeomorphic to Y (Corollary 1.7, [1, Ch. IV, Ex.34]). Q.E.D.

Definition 2.5 [10], [11]. A uniform space uX is called *coz – fine*, if each *coz* – mapping $f : uX \to vY$ is uniformly continuous.

Theorem 2.6. [10], [11]. *For a uniform space uX he following conditions are equivalent:*

(1) uX is a *coz* – fine space;

(2) uX is a $\mathcal{M}-$ fine and proximally fine space;

(3) for every mapping f of uX into a uniform space vY, if $f : uX \to v_p Y$ is uniformly continuous, then $f : uX \to v_f Y$ is uniformly continuous, where v_f is a fine uniformity on Y;

(4) for every mapping f of uX into metrizable space vY, if $f : uX \to v_p Y$ is uniformly continuous, then $f : uX \to v_f Y$ is uniformly continuous, where v_f is a fine uniformity on Y;

(5) uX is a proximally fine Alexandroff space.

Remark 2.7. Information about of $\mathcal{M}-$fine and proximally fine uniform spaces see, for example, [10], [11], [17].

Theorem 2.8. *Let uX and vY be the first-countable coz – fine uniform spaces. Then uX is uniformly homeomorphic to vY if and only if $\beta_u X$ is homeomorphic to $\beta_v Y$.*

Proof. It follows from Theorem 2.4 and Definition 2.5.

Remark 2.9. Remind, that u – open covering α of a uniform space uX is called *coz – additive*, if $\cup\alpha' \in CZ_u$ for each $\alpha' \subset \alpha$ [10], [11].

Theorem 2.10. *For uniform space uX the following conditions are equivalent:*

(1) uX is *coz* – fine;

(2) a family $\mathcal{B}^*_{\mathcal{A}f}$ of all σ – locally finite completely *coz* – additive u – open coverings is a base of uniformity u;

(3) all locally finite *coz* – additive u – open coverings form a base of uniformity u.

Proof. $(1) \Rightarrow (2)$. Evidently, for any $\alpha, \beta \in \mathcal{B}^*_{\mathcal{A}f}$ a covering $\alpha \wedge \beta$ is σ – locally finite completely *coz* –additive u – open. Hence $\alpha \wedge \beta \in \mathcal{B}^*_{\mathcal{A}f}$.

We prove a useful lemma for the continue proof of Theorem 2.10.

Lemma 2.11. *Let $\{U_s = f_s^{-1}((0,1]) : s \in S\}$ be a point-finite u – open family, where $f_s : uX \to I_s$ be a u – function and $I_s = I$ for all $s \in S$. Then it induces a function $f = \Delta\{f_s : s \in S\} : uX \to \mathbb{F}(S)$ is coz – mapping, where $\mathbb{F}(S)$ denotes the subset of $I^S = \prod\{I_s : s \in S\}$ consisting of all points $x = (x_s : s \in S)$, that have only a finite number of non-zero coordinates x_s.*

Proof. Evidently, that $\mathbb{F}(S)$ is separable metrizable space. Then for any open set $U \subset \mathbb{F}(S)$ $f^{-1}(U) = f_s^{-1}(p_s(U))$, where $p_s : \mathbb{F}(S) \to I_s$ is obvious projection. Then $f^{-1}(U) \in CZ_u$. Q.E.D.

Let $\alpha = \{W_s : s \in S\}$ be an arbitrary σ – locally finite u – open covering. For each $i \in \mathbb{N}$ a family $\alpha_i = \{W_s : s \in S_i\}$ is locally finite u – open system. Then by Lemma 2.11 there exists *coz* – mapping $f_i : uX \to \mathbb{F}(S_i)$, where $\mathbb{F}(S_i)$ is equipped with the metric d_i defined by $d_i(x, y) = \sup\{|x_s - y_s|, s \in S\}$. Each f_i is *coz* – mapping and so it is $f = \Delta\{f_i : i \in \mathbb{N}\} : uX \to \prod\{\mathbb{F}(S_i) : i \in \mathbb{N}\}$. Hence by item (1) f is uniformly continuous with respect to the metric uniformity v on $\prod\{\mathbb{F}(S_i) : i \in \mathbb{N}\}$ and it is precompact reflection v_p. By item (4) of Theorem 2.6. f is uniformly continuous with respect to the fine uniformity v_f on $\prod\{\mathbb{F}(S_i) : i \in \mathbb{N}\}$, which is a metrizable space. Therefore fine uniformity v_f has a base consisting of all open coverings [19]. For each $s \in S_i$ a set of the form $f_s^{-1}(W)$, where W is open in I_s, is the inverse image under f, is the open set $p_i^{-1}(p_s^{-1}(W))$ of $\prod\{\mathbb{F}(S_i) : i \in \mathbb{N}\}$, where $p_i : \prod\{\mathbb{F}(S_i) : i \in \mathbb{N}\} \to \mathbb{F}(S_i)$ and $p_s : \mathbb{F}(S_i) \to I_s$ are the natural projections. A covering $\beta = \{p_i^{-1}(p_s^{-1}(W)) : s \in S, i \in \mathbb{N}\}$ is uniform with respect to the fine uniformity v_f. Let γ be open σ – locally finite star-refinement of β. Then $\gamma \in v_f$ and $f^{-1}(\gamma)$ is the open σ – locally finite covering which is star-refined to α.

$(2) \Rightarrow (3)$. It is obvious.

$(3) \Rightarrow (1)$. Let $f : uX \to vY$ is uniformly continuous mapping into a metrizable uniform space vY. Then the mapping $f : uX \to v_pY$ is also uniformly continuous. By item (3) of Lemma 1.3. and metrizability of vY we have $\mathcal{Z}_v = \mathcal{Z}_{v_p} = \mathcal{Z}(Y)$. The fine uniformity v_f of the metrizable uniform space vY has a base consisting of all open locally finite coverings, hence, $f : uX \to v_fY$ is also uniformly continuous mapping. Theorem is proved completely.

Corollary 2.12 *For a uniform space uX there exists such coz – fine uniformity u_{cf}, that $u \subset u_{cf}$ and $u_\omega^z \subset u_{cf}$.*

Proof. A uniformity u has a base of some family of σ – uniformly discrete completely *coz* – additive u – open coverings, hence $u \subset u_{cf}$. Every countable u – open covering is a σ – locally finite u – open covering, hence $u_\omega^z \subset u_{cf}$. Q.E.D.

Corollary 2.13 *Every Cauchy z_u – ultrafilter with respect to uniformity u_{cf} is countably centered.*

Proof. It follows from $u_\omega^z \subset u_{cf}$.

Corollary 2.14. *The completion μ_uX of the uniform space uX with respect to the uniformity u_{cf} is contained in the Wallman realcompactification v_uX, i.e. $\mu_uX \subset v_uX \subset \beta_uX$.*

Proof. It follows immediately from Corollary 2.13.

Corollary 2.15. *Let w be a uniformity of completion μ_uX. Then $\beta_v(\mu_uX) = \beta_uX$.*

Proof. It follows immediately from Corollary 2.14.

Theorem 2.16. *Let uX and vY be first-countable coz – fine uniform spaces. Then $\propto_uX$ is uniformly homeomorphic to μ_vY if and only if β_uX is homeomorphic to β_vY.*

Proof. Let w and w' be a uniformity of completions μ_uX and μ_vY, respectively. Then $\beta_w(\mu_uX) = \beta_uX$ and $\beta_{w'}(\mu_vY) = \beta_vY$ and β_uX is homeomorphic to β_vY, if μ_uX is uniformly homeomorphic to μ_vY.

Conversely, from homeomorphity of β_uX and β_vY it follows uniform homeomorphity of uX and vY (see Theorem 2.8). Then completions μ_uX and μ_vY are uniformly homeomorphic to each other. Q.E.D.

Corollary 2.17 *Let uX and vY be complete the first-countable coz – fine uniform spaces. Then uX is uniformly homeomorphic to vY if and only if β_uX is homeomorphic to β_vY.*

Proof. It follows immediately from Theorem 2.16.

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