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ФУНКТОР P_n И ПАРАКОМПАКТНЫЕ *P*-ПРОСТРАНСТВА

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Основной результат: свойства пространств $P_n(X) \in C$ и $X^n \in C$ эквивалентны, где $P_n -$ функтор вероятностных мер с конечными носителями, а $X -$ паракомпактное *p*-пространство. Все пространства предполагаются нормальными, все отображения – непрерывными.

Ключевые слова: функтор; паракомпактное *p*-пространство; категория; вероятностная мера; носитель.

FUNCTOR P_n AND PARACOMPACT *P*-SPACES

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The main result is: the properties of spaces $P_n(X) \in C$ and $X^n \in C$ are equivalent where $P_n -$ is the functor of probability measures with finite supports, and $X -$ is a paracompact *p*-space. All spaces are assumed to be normal, and all mappings are continuous.

Keywords: functor; paracompact p-space; category; probability measure; support.

Definition. 1 *A space X is called C – (notation: $X \in C$) if for every sequence $u_i \in \mathrm{cov}(X)$, $i \in \mathbb{N}$ of open coverings of a space X there exists a sequence v_i of disjoint families of open subsets of X, such that v_i refines u_i and $\bigcup\{v_i : i \in \mathbb{N}\} \in \mathrm{cov}(X)$.*

Definition. 2 *A covariant functor $\mathcal{F} : \mathrm{Comp} \to \mathrm{Comp}$, acting in the category of compact spaces and their continuous mappings is called a normal functor if it*

1) preserves a point and the empty set;
2) preserves the weight of infinite compacta;
3) is monomorphic;
4) is epimorphic;
5) is continuous;
6) preserves intersections;
7) preserves inverse images.

For a functor $\mathcal{F}$ preserving intersections and for any element $a \in \mathcal{F}(X)$ we define its *support* supp a as $\mathrm{supp}\, a = \bigcap\{Y \subset X : Y \text{ is closed and } a \in \mathcal{F}_X(Y)\}$. For a normal functor $\mathcal{F}$, we define its subfunctor $\mathcal{F}_n$, $n \in \mathbb{N}$ in the following way: $\mathcal{F}_n(X) = \{a \in \mathcal{F}(X) : |\mathrm{supp}\, a| \le n\}$. It is easy to check that $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$ and thus $\mathcal{F}_n$ is a normal functor in the category Comp . In this case, the functor $\mathcal{F}_1$ is isomorphic to the identity functor Id . The functor *P* of probability measures and its subfunctors P_n are normal functors ([1], Chapter VII).

For a normal functor $\mathcal{F}$ and a compactum *X*, Basmanov [2] defined a sequence of mappings

$$\pi = \pi_n = \pi_{n,X} : C(n, X) \times \mathcal{F}(n) \to \mathcal{F}_n(X) \tag{1}$$

by the equality

$$\pi(\xi, a) = \mathcal{F}(\xi)(a). \tag{2}$$

Mappings (1) can be defined for an arbitrary Tychonoff space. To do this, we need only define the space $\mathcal{F}_n(X)$. Let

$$\mathcal{F}_n(X)=\left\{a\in\mathcal{F}_n\left(\beta X\right):\operatorname{supp} a\subset X\right\} \tag{3}$$

The identity embedding $\mathcal{F}_n(X)\subset\mathcal{F}_n(\beta X)$ induces a topology on the set $\mathcal{F}_n(X)$. Equation (3) defines a functor $\mathcal{F}_n$:Tych $\to$ Tych, which have all properties of normal functor. Nevertheless, mappings (1), generally speaking, are not closed. But we prove the following statement.

Proposition. 3 *The mapping*

$\pi=\pi_{n,X}:\pi_{n,X}^{-1}(\mathcal{F}_n(X)\setminus\mathcal{F}_{n-1}(X))\to\mathcal{F}_n(X)\setminus\mathcal{F}_{n-1}(X)$ *is closed for any Tychonoff space X.*

To prove this statement, it is enough to consider the embedding $X\subset\beta X$ and show that

$$\pi_{n,X}^{-1}(\mathcal{F}_n(X)\setminus\mathcal{F}_{n-1}(X))=\pi_{n,\beta X}^{-1}(\mathcal{F}_n(X)\setminus\mathcal{F}_{n-1}(X)). \tag{4}$$

The proof of equality (4) splits into several lemmas, in which the set $\mathcal{F}_n(X)\setminus\mathcal{F}_{n-1}(X)$ is denoted by *Y* and equation (2) is multiply applied.

Lemma. 4 *If* $\mathcal{F}(\xi)(a)=b\in Y$ *, then* $\operatorname{Im}\xi=\operatorname{supp} b$.

Proof. Let $\operatorname{supp} b=\{x_0,\dots,x_{n-1}\}$. Since $\mathcal{F}$ preserves supports, we have $\xi(\operatorname{supp} a)=\operatorname{supp}\mathcal{F}(\xi)(a)=$ $=\operatorname{supp} b=\{x_0,\dots,x_{n-1}\}$. Therefore, $|\operatorname{supp} a|\geq n,$ i. e. $\operatorname{supp} a=\{0,1,\dots,n-1\}$. Therefore, $\operatorname{Im}\xi=\xi(\operatorname{supp} a)=$ $=\operatorname{supp} b.$

Lemma. $\pi_{n,\beta X}\,|\,C(n,X)\times\mathcal{F}(n)=\pi_{n,X}$ **5**.

Proof. Have $\pi_{n,\beta X}(\xi,a)=\mathcal{F}(\xi)(a)=({}_{since}\,\xi\in C(n,X))=\pi_{n,X}(\xi,a).$

The next results folows from Lemma 1:

Lemma. $\pi_{n,\beta X}^{-1}(Y)\subset C(n,X)\times\mathcal{F}(n)$ **6**.

Proof of equality (4). It suffices to check the inclusion $\supset$. Let $(\xi,a)\in\pi_{n,\beta X}^{-1}(Y)$. So, $\xi\in C(n,X)$ by Lemma 3. But then $(\xi,a)\in\pi_{n,X}^{-1}(Y)$ by Lemma 2.

Recall that a mapping $f:X_1\to X_2$ is called *local homeomorphism*, if for every point $x\in X_1$ there exists a neighborhood Ox such that the mapping $f:Ox\to f(Ox)$ is a homeomorphism onto an open subset of X_2.

Proposition. 7 *For any normal functor* $\mathcal{F}:\mathrm{Comp}\to\mathrm{Comp}$ *and any Tychonoff space* X*, the mapping*

$\pi_{n,X}:\pi_{n,X}^{-1}(\mathcal{F}_n(X)\setminus\mathcal{F}_{n-1}(X))\to\mathcal{F}_n(X)\setminus\mathcal{F}_{n-1}(X)$ *is a local homeomorphism. Thus* $|\pi_{n,X}^{-1}(b)|=n!$ *for any* $b\in\mathcal{F}_n(X)\setminus\mathcal{F}_{n-1}(X)$.

Proof. Let $(\xi,a)\in\pi_{n,X}^{-1}(Y)$, where $Y=\mathcal{F}_n\left(X\right)\setminus\mathcal{F}_{n-1}(X)$ and let $\pi_{n,X}(\xi,a)=b$. The support of the element b consists of n points $x_0,\dots,x_{n-1}$. We can assume that $\xi(i)=x_i$. Next, since $b\in Y$, we have $a\in\mathcal{F}_n(n)\setminus\mathcal{F}_{n-1}(n)n$. Then

$$\pi_{n,X}^{-1}(b)=\{(\xi\circ\sigma,\ \mathcal{F}(\sigma^{-1})(a)):\sigma\in S_n\}, \tag{5}$$

Where S_n is the symmetric group of permutations of the set $n=\{0,1,\dots,n-1\}$. Indeed, let $(\xi',a')\in\pi_{n,X}^{-1}(b)$. Then from Lemma 1 it follows that ξ' is a bijection of the set *n* onto $\{x_0,\dots,x_{n-1}\}$ and therefore $\xi'=\xi\times\sigma$ for some $\sigma\in S_n$. Next,

$$\mathcal{F}(\xi)(a)=b=\mathcal{F}(\xi')(a')=\mathcal{F}(\xi\circ\sigma)(a')=\mathcal{F}(\xi)(\mathcal{F}(\sigma)(a')),\ \text{ i.e. }\ \mathcal{F}(\xi)(a)=\ \mathcal{F}(\xi)\Big(\mathcal{F}\big(\sigma\big)\big(a'\big)\Big). \tag{6}$$

But $\xi:n\to\{x_0,\dots,x_{n-1}\}$ is a monomorphism and $\mathcal{F}$ preserves monomorphisms. Therefore, $a=\mathcal{F}(\sigma)(a')$, i.e. $a'=\mathcal{F}(\sigma^{-1})(a)$. Thus, equation (5) together with the last assertion of Proposition 2 are established.

Consider any neighborhoods $Ox_0,\dots,Ox_{n-1}$ of the points $x_0,\dots,x_{n-1}$ such that any two neighborhoods have nonempty intersection. Let $U_\sigma=Ox_{\sigma(0)}\times\ldots\times Ox_{\sigma(n-1)}\times(\mathcal{F}(n)\setminus\mathcal{F}_{n-1}(n))$. Identifying the monomorphism ξ with

Imξ, we find that the set U_σ is a neighborhood, in the space $C(n,X)\times\mathcal{F}(n)$, of any pair (ξ,a), where $a\in\mathcal{F}(n)\setminus\mathcal{F}_{n-1}(n)$ and $\mathrm{Im}\,\xi\cap Ox_{\sigma(i)}\neq\varnothing$ for every $i=0,\ldots,n-1$. Therefore, equality (5) implies that

$$\pi_{n,X}^{-1}(\pi_{n,X}(U))=U, \tag{7}$$

where $U=\cup\{U_\sigma:\sigma\in S_n\}$. From Proposition 1 and equality (6), the openness of the set $\pi_{n,X}(U)$ follows. Further, the components U_σ in equation (7) are pairwise disjoint and, therefore, are clopen in *U*. It remains to show that the mappings

$$\pi_{n,X}:U_\sigma\to\pi_{n,X}(U) \tag{8}$$

are bijections, since these mappings are closed by Proposition 1. But, according to (5), the inverse image $\pi_{n,X}^{-1}(b)$ consists of $n!$ points for every $b\in Y$. Therefore, the mappings from (8) are bijections, since we will prove that all of them are surjective. But the latter also follows from (5).

Measures on paracompact *p*-spaces

Theorem. 8 *Let X be a paracompact* $p-$ *space. Then* $P_n(X)\in C\Leftarrow X^n\in C$.

Proof. Let's start with the implication $\Rightarrow$. We use the induction on *n*. When $n=1$ the statement is evident. To make the inductive transition $n-1\to n$, it suffices to show that $X^n\times P(n)\in C$. For this, in turn, it is enough to check that $\pi_n^{-1}(P_{n-1}(X))\in C$,

$$\pi_n^{-1}Z\in C \tag{9}$$

for any closed set $Z\subset P_n(X)\setminus P_{n-1}(X)$. Property (9) follows from a theorem of Hattori-Yamada about a closed inverse image of $C-$ spaces [3], paracompactness of *Z*, and the statement that the mapping $\pi_n:\pi_n^{-1}Z\to Z$ is a local homeomorphism according to Proposition 2.

Now, for $1\le i<j\le n$ we define the set $X_{ij}\subset X^n$ as follows. Assume that $X^n=X_1\times\ldots\times,X_n$, where X_k are copies of the space *X*. Denote by Δ_{ij} the diagonal of the square $X_i\times X_j$ and put $X_{ij}=\Delta_{ij}\times\prod\{X_k:k\neq i,\ k\neq j\}$. Then the space X_{ij} is homeomorphic to X^{n-1} and so is a $C-$space by the induction hypothesis, since $P_{n-1}(X)$ is closed in $P_n(X)$.

Put $Y=(\cup\{X_{ij}:1\le i<j\le n\})\times P(n)$. The space *Y* is the union of a finite number of closed subspaces $X_{ij}\times P(n)$, each of them being a $C-$ space by Hattori-Yamada theorem on the product of $C-$ spaces [3]. Therefore, $Y\in C$. To complete the proof, it is enough to check that $\pi_n^{-1}(P_{n-1}(X))\subset Y$. But if $\pi_n(\xi,\mu)=P(\xi)(\mu)\in P_{n-1}(X)$, then there exist two coordinates x_i and x_j of a point $\xi=(x_1,\ldots,,x_n)$ such that $x_i=x_j$ and, consequently, $\xi\in X_{ij}$.

Now let us check the implication $\Leftarrow$. If $X^n\in C$, then $X^n\times P(n)\in C$ by the above mentioned theorem of Hattori-Yamada. The condition $P_n(X)\in C$ can be checked by induction on *n*. For the implementation of inductive transition $n-1\to n$, it is enough to show that $Z\in C$ for any subset $Z\subset P_n(X)\setminus P_{n-1}(X)$ closed in $P_n(X)$. But $Z=\pi_{n,X}(\pi_{n,X}^{-1}(Z))$ and the mapping $\pi_{n,X}:\pi_{n,X}^{-1}(Z)\to Z$ is closed due to Proposition 1, and it is finite-fold according to equality (5). Therefore $Z\in C$ by Theorem 3.5 of [4].

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